THE DISTINCTNESS AND GENERATING FIELDS OF TWISTED KLOOSTERMAN SUMS

SHENXING ZHANG

Abstract. We use the Kloosterman sheaves constructed by Fisher to show when two Kloosterman sums differ a $(q-1)$-th root of unity, and use p-adic analysis to prove the non-vanishing of the Kloosterman sums. Then we can determine the generating fields by these results.

Contents

1. Introduction

1.1. **Background.** Let p be a prime number, $q = p^d$ a power of p, and $\mathbb{F}_q$ the field with q elements. Let $\psi : \mathbb{F}_p \to \mu_p$ be a fixed non-trivial additive character. For $\boldsymbol{\chi} = \{\chi_1, \ldots, \chi_n\}$ an unordered n-tuple of multiplicative characters $\chi_i : \mathbb{F}_q^\times \to \mu_{q-1}$ and $a \in \mathbb{F}_q^\times$, define the *Kloosterman sum* as

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \sum_{\substack{x_1 \cdots x_n = a \\ x_i \in \mathbb{F}_q}} \chi_1(x_1) \cdots \chi_n(x_n) \psi\big(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1 + \cdots + x_n)\big).$$

Clearly it lies in $\mathbb{Z}[\mu_{p(q-1)}]$.

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When $\boldsymbol{\chi} = \mathbf{1} = \{1, \cdots, 1\}$ is trivial, the distinctness of Kloosterman sums is studied by many peoples. It's easy to see that

$$a, b \text{ conjugate} \implies \mathrm{Kl}_n(\psi, \mathbf{1}, q, a) = \mathrm{Kl}_n(\psi, \mathbf{1}, q, b).$$

It's a conjecture (Ref conjecture) that the converse is true when $p \geq nd$, see [Fis92, Remark 4.28(2)]. This is true when $p > (2n^{2d}+1)^2$ in [Fis92], or $p \geq (d-1)n+2$ and p does not divide a certain integer in [Wan95, Theorem 1.3]. In these cases, one can obtain that the algebraic degree of $\mathrm{Kl}_n(\psi, \mathbf{1}, q, a)$ is $(p-1)/(p-1, n)$. When $\mathrm{Tr}(a) \neq 0$ or the Ref conjecture holds for $\mathbb{F}_q$, the algebraic degree is given in [Wan95] and [KRV11].

1.2. **Notations and main results.** In this article, we will study the twisted version. More precisely, we will study the distinctness of Kloosterman sums up to $(q-1)$-th roots of unity, the non-vanishing and the generating fields of Kloosterman sums.

Let m be an integer prime to p, such that $\chi_i^m = 1$ for all i. For any integer $w \in \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, any multiplicative character Λ and $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, denote by

$$\boldsymbol{\chi}^w = \{\chi_1^w, \cdots, \chi_n^w\}, \quad \boldsymbol{\chi}\Lambda = \{\chi_1\Lambda, \cdots, \chi_n\Lambda\}, \quad \boldsymbol{\chi} \circ \sigma = \{\chi_1 \circ \sigma, \cdots, \chi_n \circ \sigma\}$$

and $\prod \boldsymbol{\chi} = \chi_1 \cdots \chi_n$ for abbreviations. The Galois group

$$\mathrm{Gal}\big(\mathbb{Q}(\mu_{pm})/\mathbb{Q}\big) = \Big\{\sigma_t\tau_w \mid t \in (\mathbb{Z}/p\mathbb{Z})^\times, w \in \big(\mathbb{Z}/m\mathbb{Z}\big)^\times\Big\},$$

where

$$\sigma_t(\zeta_p) = \zeta_p^t, \quad \sigma_t(\zeta_m) = \zeta_m,$$
$$\tau_w(\zeta_p) = \zeta_p, \quad \tau_w(\zeta_m) = \zeta_m^w,$$

for any $\zeta_p \in \mu_p$, $\zeta_{q-1} \in \mu_m$. We will take m to be $q-1$, or

$$c = \text{the least common multiplier of the orders of } \chi_i. \tag{1.1}$$

Definition 1.1. The n-tuple $\boldsymbol{\chi}$ is called *Kummer-induced* if there exsists a non-trivial character Λ such that $\boldsymbol{\chi} = \boldsymbol{\chi}\Lambda := \{\chi_1\Lambda, \ldots, \chi_n\Lambda\}$ as unordered n-tuples. In this case, $\prod \boldsymbol{\chi} = \prod(\boldsymbol{\chi}\Lambda) = \Lambda^n \prod \boldsymbol{\chi}$ and thus $\Lambda^n = 1$.

A basic observation tells

$$\sigma_t\tau_w \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \prod \boldsymbol{\chi}(t)^{-w} \mathrm{Kl}_n(\psi, \boldsymbol{\chi}^w, q, at^n).$$

To obtain its generating field, we need to know when two Kloosterman sums are same up to a $(q-1)$-th root of unity.

In Section 2, we will recall the construction of Kloosterman sheaves by Fisher and follow his method to show the following theorem. Denote by Λ_2 the non-trivial quadratic character on $\mathbb{F}_q^\times$.

Theorem 1.2. *Let $a, b \in \mathbb{F}_q^\times$ and let $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$ be n-tuples of multiplicative characters. Assume that $\boldsymbol{\chi}, \boldsymbol{\rho}$ are not Kummer-induced and neither of them is of type $(\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$. If $p > (2n^{2d}+1)^2$ and*

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for some $\lambda \in \mu_{q-1}$, then there exists $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and a multiplicative character η, such that $b = \sigma(a)$ and $\boldsymbol{\rho} = \eta \cdot (\boldsymbol{\chi} \circ \sigma^{-1})$ as unordered tuples. Moreover, either both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$.

In Section 3, we will prove the non-vanishing of Kloosterman sum by p-adic analysis. We need the following condition on $\boldsymbol{\chi}$:

$$\text{For any } i, j,\ \chi_i = \chi_j \text{ if } \chi_i^n = \chi_j^n. \tag{1.2}$$

That's to say, $\chi_i = \chi_j$, or $\chi_i\chi_j^{-1}$ is not a character of order dividing n. Denote by

$$C_{\boldsymbol{\chi}} = \max_{i,j} \operatorname{lcm}\big(\operatorname{ord}(\chi_i), \operatorname{ord}(\chi_j)\big) \tag{1.3}$$

the supremum of least common multipliers of the orders of any two characters in $\boldsymbol{\chi}$.

Theorem 1.3. *If $p > (3n-1)C_{\boldsymbol{\chi}} - n$ and $\boldsymbol{\chi}$ satisfies (1.2), then $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$ is nonzero.*

In Section 4, we will discuss the generating fields and give several examples.

Theorem 1.4. *If $p > \max\big\{(2n^{2d}+1)^2, (3n-1)C_{\boldsymbol{\chi}} - n\big\}$ and $\boldsymbol{\chi}$ satisfies (1.2), then $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t\tau_w$ such that there exists an integer β and a character η satisfying*

$$t = \lambda a_1^{\beta}, \lambda^{n_1} = 1,\ \boldsymbol{\chi}^w = \eta\boldsymbol{\chi}^{q_1^{\beta}},\ \eta(a) = \prod \boldsymbol{\chi}^w(t).$$

Here $n_1 = (n, p-1)$, $q_1 = \#\mathbb{F}_p(a^{(p-1)/n_1})$ and $a_1 \in \mathbb{F}_p^{\times}$ such that $a_1^{n/n_1} = \mathbf{N}_{\mathbb{F}_{q_1}/\mathbb{F}_p}(a^{(1-p)/n_1}) = a^{(1-q_1)/n_1}$.

2. Kloosterman sheaves constructed by Fisher

2.1. Kloosterman sheaves. Let $\ell \neq p$ be a prime. We fix an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. Then the additive and multiplicative character ψ, χ_i can take value both in $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$.

Deligne in [Del77, Theorem 7.8] and Katz in [Kat88, Theorem 4.11] defined the Kloosterman sheaf

$$\mathcal{K}\mathrm{l} = \mathcal{K}\mathrm{l}_{n,q}(\psi, \boldsymbol{\chi})$$

on $\mathbb{G}_m \otimes \mathbb{F}_q$, with the following properties:

(1) $\mathcal{K}\mathrm{l}$ is lisse of rank n and pure of weight $n-1$.
(2) For any $a \in \mathbb{F}_q^{\times}$, $\mathrm{Tr}\big(\mathrm{Frob}_a, \mathcal{K}\mathrm{l}_{\overline{a}}\big) = (-1)^{n-1}\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)$.
(3) $\mathcal{K}\mathrm{l}$ is tame at 0.
(4) $\mathcal{K}\mathrm{l}$ is totally wild with Swan conductor 1 at ∞. So all ∞-breaks are $1/n$.

Remark 2.1. When $\boldsymbol{\chi}$ is not Kummer-induced, $\mathcal{K}\mathrm{l}$ is not geometrically Kummer-induced. That's to say, $\mathcal{K}\mathrm{l}$ is not of type $(t \mapsto t^N)_*\mathcal{F}$ for some positive integer $N > 1$ and some lisse sheaf $\mathcal{F}$ on $\mathbb{G}_m \otimes \overline{\mathbb{F}}_q$. See [Fis92, Theorem 2.9].

2.2. Fisher's descent. In [Fis92, §3], Fisher gave a descent of Kloosterman sheaves along an extension of finite fields. For any $a \in \mathbb{F}_q^{\times}$, he defined a lisse sheaf $\mathcal{F}_a(\boldsymbol{\chi})$ on $\mathbb{G}_m \otimes \mathbb{F}_p$, such that

(1) $\mathcal{F}_a(\boldsymbol{\chi})|\mathbb{G}_m \otimes \mathbb{F}_q = \bigotimes_{\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)} \big(t \mapsto \sigma(a)t^n\big)^* \mathcal{K}\mathrm{l}_n(\psi \circ \sigma^{-1}, \boldsymbol{\chi} \circ \sigma^{-1})$.
(2) $\mathcal{F}_a(\boldsymbol{\chi})$ is lisse of rank n^d and pure of weight $d(n-1)$.
(3) For any $t \in \mathbb{F}_p^{\times}$, $\mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}_a(\boldsymbol{\chi})_{\overline{t}}\big) = (-1)^{(n-1)d}\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, at^n)$.
(4) $\mathcal{F}_a(\boldsymbol{\chi})$ is tame at 0 and its ∞-breaks are at most 1.

Assume that $p > 2n+1$ and $\boldsymbol{\chi}$ is not Kummer-induced. Then $\mathcal{F}_a(\boldsymbol{\chi})$ has a highest weight with multiplicity one. Thus it has a subsheaf $\mathcal{G}_a(\boldsymbol{\chi})$ such that, as representations of the Lie algebra $\mathfrak{g}(\mathcal{F}_a(\boldsymbol{\chi}))$, $\mathcal{G}_a(\boldsymbol{\chi})$ is the irreducible sub-representation with highest weight. Moreover, it is geometrically irreducible and occurs exactly once in $\mathcal{F}_a(\boldsymbol{\chi})$ over $\mathbb{G}_m \otimes \overline{\mathbb{F}}_p$. See [Fis92, Proposition 4.18].

The additive character ψ can be viewed as a character on $\mathbb{F}_p$-points of $\mathbb{B} = \mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p}\mathbb{G}_a$. It gives a rank one lisse sheaf L_ψ on $\mathbb{B}$ constructed from the Lang torsor as in [Kat88, §4.3]. We still denote by L_ψ its restriction on $\mathbb{B}^\times$. Denote by $\mathcal{L}_\psi$ its pull-back along $\mathbb{G}_m \otimes \mathbb{F}_p \to \mathbb{B}^\times, t \mapsto t \otimes 1$. For the multiplicative character χ, we can define $\mathcal{L}_\chi$ similarly. Then for $t \in \mathbb{F}_p^\times$,

$$\mathrm{Tr}\big(\mathrm{Frob}_t, (\mathcal{L}_\psi)_{\bar{t}}\big) = \psi(t), \quad \mathrm{Tr}\big(\mathrm{Frob}_t, (\mathcal{L}_\chi)_{\bar{t}}\big) = \chi(t).$$

2.3. **Distinctness.** We will consider when

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for some $\lambda \in \mu_{q-1}$. The argument is almost the same as in [Fis92], while $\lambda = 1$ in his paper. So we will only show the difference.

Lemma 2.2. *Let $\mathcal{F}, \mathcal{F}'$ be lisse sheaves on $\mathbb{G}_m \otimes \mathbb{F}_p$ of same rank r and pure of the same weight w. Assume that there is a root of unity λ such that for any $t \in \mathbb{F}_p^\times$, we have*

$$\mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}\big) = \lambda \mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}'_{\bar{t}}\big).$$

Let $\mathcal{G}$ be a geometrically irreducible sheaf of rank s on $\mathbb{G}_m \otimes \mathbb{F}_p$, pure of weight w, such that $\mathcal{G} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ occurs exactly once in $\mathcal{F} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$. Then $\mathcal{G} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ occurs at least once in $\mathcal{F}' \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$, provided that $p > [2rs(M_0 + M_\infty) + 1]^2$, where M_η is the largest η-break of $\mathcal{F} \oplus \mathcal{F}'$.

Proof. See [Fis92, Lemma 4.9]. Assume that $\mathcal{G} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ does not occur in $\mathcal{F}' \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$. We reduce to the case $w = 0$ by a twist. We have

$$\begin{aligned}&\mathrm{Tr}\big(\mathrm{Frob}_t, (\mathcal{G}^\vee \otimes \mathcal{F})_{\bar{t}}\big) = \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{G}^\vee_{\bar{t}}) \cdot \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}) \\ =&\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{G}^\vee_{\bar{t}}) \cdot \lambda \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}'_{\bar{t}}) = \lambda \mathrm{Tr}\big(\mathrm{Frob}_t, (\mathcal{G}^\vee \otimes \mathcal{F}')_{\bar{t}}\big).\end{aligned}$$

Applying the Lefschetz Trace Formula to $\mathcal{G}^\vee \otimes \mathcal{F}$ and $\mathcal{G}^\vee \otimes \mathcal{F}'$, we have

$$\sum_{i=0}^{2} (-1)^i \mathrm{Tr}\big(\mathrm{Frob}_p, \mathrm{H}^i_c(\mathcal{G}^\vee \otimes \mathcal{F})\big) = \lambda \sum_{i=0}^{2} (-1)^i \mathrm{Tr}\big(\mathrm{Frob}_p, \mathrm{H}^i_c(\mathcal{G}^\vee \otimes \mathcal{F}')\big).$$

Note that $\mathrm{H}^0_c = 0$,

$$\mathrm{H}^2_c(\mathcal{G}^\vee \otimes \mathcal{F}) = \mathrm{Hom}(\mathcal{G}, \mathcal{F})_{\pi_1^{\mathrm{geom}}(\mathbb{G}_m \otimes \overline{\mathbb{F}}_p)}(-1)$$

is one-dimensional, pure of weight 2,

$$\mathrm{H}^2_c(\mathcal{G}^\vee \otimes \mathcal{F}') = \mathrm{Hom}(\mathcal{G}, \mathcal{F}')_{\pi_1^{\mathrm{geom}}(\mathbb{G}_m \otimes \overline{\mathbb{F}}_p)}(-1) = 0,$$

$\mathrm{H}^1_c(\mathcal{G}^\vee \otimes \mathcal{F}), \mathrm{H}^1_c(\mathcal{G}^\vee \otimes \mathcal{F})$ are mixed of weight ≤ 1 by Weil II [Del80]. Therefore

$$\begin{aligned} p &= \big|\mathrm{Tr}\big(\mathrm{Frob}_p, \mathrm{H}^2_c(\mathcal{G}^\vee \otimes \mathcal{F})\big)\big| \\ &= \big|\mathrm{Tr}\big(\mathrm{Frob}_p, \mathrm{H}^1_c(\mathcal{G}^\vee \otimes \mathcal{F})\big) - \lambda \mathrm{Tr}\big(\mathrm{Frob}_p, \mathrm{H}^1_c(\mathcal{G}^\vee \otimes \mathcal{F}')\big)\big| \\ &\leq \sqrt{p}\big(h^1_c(\mathcal{G}^\vee \otimes \mathcal{F}) + h^1_c(\mathcal{G}^\vee \otimes \mathcal{F}')\big). \end{aligned}$$

By Euler-Poincaré formula,

$$h^1_c(\mathcal{G}^\vee \otimes \mathcal{F}) = \mathrm{Sw}_0(\mathcal{G}^\vee \otimes \mathcal{F}) + \mathrm{Sw}_\infty(\mathcal{G}^\vee \otimes \mathcal{F}) + 1$$
$$h^1_c(\mathcal{G}^\vee \otimes \mathcal{F}') = \mathrm{Sw}_0(\mathcal{G}^\vee \otimes \mathcal{F}') + \mathrm{Sw}_\infty(\mathcal{G}^\vee \otimes \mathcal{F}').$$

Therefore $p \leq \big(2rs(M_0 + M_\infty) + 1\big)^2$. □

Corollary 2.3. *Let $a, b \in \mathbb{F}_q^\times$ and let $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$ be n-tuples of multiplicative characters $\chi_i, \rho_j : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$. Assume that $p > (2n^{2d} + 1)^2$, $\boldsymbol{\chi}$ is not Kummer-induced and*

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for a fixed root of unity $\lambda \in \mu_{q-1}$. Then $\mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ occurs at least once in $\mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$.

Proof. See [Fis92, Corollary 4.20]. Denote by

$$\mathcal{F} = \mathcal{F}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}},\ \mathcal{F}' = \mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}},\ \mathcal{G} = \mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}.$$

For $t \in \mathbb{F}_p^\times$, we have $\sigma_t \lambda = \lambda$ and thus

$$(-1)^{(n-1)d} \mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}_{\bar{t}}\big) = \prod \overline{\boldsymbol{\chi}}(t) \cdot \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, at^n) = \sigma_t\big(\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)\big)$$
$$= \lambda \sigma_t\big(\mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)\big) = \lambda \prod \overline{\boldsymbol{\rho}}(t) \cdot \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, bt^n) = (-1)^{(n-1)d} \lambda \mathrm{Tr}\big(\mathrm{Frob}_t, \mathcal{F}'_{\bar{t}}\big).$$

Apply Lemma 2.2 to these sheaves with $r = s = n^d, M_0 = 0, M_\infty \leq 1$, the result then follows. □

Proof of Theorem 1.2. By our assumptions, the Kloosterman sheaves $\mathcal{K}\mathrm{l}_n(\psi, \boldsymbol{\chi})$ and $\mathcal{K}\mathrm{l}_n(\psi, \boldsymbol{\rho})$ are not Kummer-induced by [Fis92, Theorem 2.9]. If the connected geometric monodromy group $G_{\mathrm{geom}}\big(\mathcal{K}\mathrm{l}_n(\psi, \boldsymbol{\chi})\big)^\circ = \mathrm{SO}(4)$, by [Fis92, Proposition 2.10], $n = 4$ and there is a multiplicative character η such that $\overline{\boldsymbol{\chi}} = \boldsymbol{\chi}\eta$ as unordered 4-tuples and $\prod \boldsymbol{\chi} = \Lambda_2 \eta^{-2}$. Since $\boldsymbol{\chi}$ is not Kummer-induced, we have $\boldsymbol{\chi} = (\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$ for some ξ_1, ξ_2. This contradicts to our assumptions. Thus $G_{\mathrm{geom}}\big(\mathcal{K}\mathrm{l}_n(\psi, \boldsymbol{\chi})\big)^\circ \neq \mathrm{SO}(4)$. Similarly, $G_{\mathrm{geom}}\big(\mathcal{K}_n(\psi, \boldsymbol{\rho})\big)^\circ \neq \mathrm{SO}(4)$.

As showned in [Fis92, Theorem 4.22], we have

$$\mathcal{G}_a(\boldsymbol{\chi}) \hookrightarrow \mathcal{F}_b(\boldsymbol{\rho}), \quad \mathcal{G}_b(\boldsymbol{\rho}) \hookrightarrow \mathcal{F}_a(\boldsymbol{\chi}),$$

by applying Corollary 2.3 twice. By following Fisher's argument step by step, there are $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and a multiplicative character η, such that $b = \sigma(a)$ and $\boldsymbol{\rho} = \eta \cdot (\boldsymbol{\chi} \circ \sigma^{-1})$ as unordered tuples. Finally,

$$\mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b) = \eta(b) \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a).$$

Hence both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$. □

Remark 2.4. In [Fis92, Corollary 4.27], Fisher showed that if $p > (2n^{4d} + 1)^2$ and

$$|\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a)| = |\mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)|,$$

then $b = \sigma(a), \boldsymbol{\rho} = \eta \cdot (\boldsymbol{\chi} \circ \sigma^{-1})$, or $b = (-1)^n \sigma(a), \boldsymbol{\rho} = \eta \cdot (\boldsymbol{\chi} \circ \sigma^{-1})$.

Corollary 2.5. *Keeping the hypotheses of Theorem 1.2. Assume that $\boldsymbol{\chi}$ is defined over $\mathbb{F}_p$, that's to say, $\boldsymbol{\chi} = \boldsymbol{\chi}_0 \circ \mathbf{N}_{\mathbb{F}_q/\mathbb{F}_p}$ for some n-tuple $\boldsymbol{\chi}_0$ of characters on $\mathbb{F}_p^\times$. Then*

$$\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, b), \quad \lambda \in \mu_{q-1}$$

if and only if $b = \sigma(a)$ for some $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, and $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, b)$.

Proof. In this case, we have $\boldsymbol{\chi} = \eta\boldsymbol{\chi}$ and then $\eta = 1$. The result then follows easily. □

3. The non-vanishing of Kloosterman sums

We will prove Theorem 1.3 in this section.

Proof of Theorem 1.3. Let $\mathfrak{p}$ be a prime above p in $\mathbb{Q}(\mu_{q-1})$ and $\mathfrak{P}$ the unique prime above $\mathfrak{p}$ in $\mathbb{Q}(\mu_{(q-1)p})$. Let v the normalized $\mathfrak{P}$-adic valuation. Once we fix an isomorphism from $\mathbb{F}_q$ to the residue field of $\mathfrak{p}$, the Teichmüller lifting of the residue map at $\mathfrak{p}$ is a primitive character of $\mathbb{F}_q^\times$, which is denoted by ω. Denote by

$$g(m) = \sum_{t\in\mathbb{F}_q^\times} \omega^{-m}(t)\psi\big(\mathrm{Tr}(t)\big)$$

the Gauss sum. Then the Stickelberger's congruence theorem tells that

$$v\big(g(m)\big) = \sum_{j=0}^{d-2} m_j, \tag{3.1}$$

where

$$0 \le m \le q-2, \quad m = \sum_{j=0}^{d-1} m_j p^j,\ 0 \le m_j \le p-1,$$

see [Sti90] or [Was97, Chap. 6].

For any $1 \le i \le n$, there is s_i such that $\chi_i = \omega^{-s_i}$. Note that

$$\sum_{m=0}^{q-2} \omega^{-m}(x) = \begin{cases} q-1, & \text{if } x = 1; \\ 0, & \text{if } x \ne 1. \end{cases}$$

Take $x = x_1 \cdots x_n a^{-1}$, we have

$$(q-1)\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \sum_{m=0}^{q-2} \omega^m(a) \prod_{i=1}^n g(m+s_i).$$

There is a unique m such that $v\big(\prod_{i=1}^n g(m+s_i)\big)$ is minimal by Proposition 3.1. Thus the valuation of the Kloosterman sum is finite and the Kloosterman sum is nonzero. □

We may assume that $1 \le s_i \le q-1$ (notice the bound). Write

$$s_i = \sum_{j=0}^{d-1} s_{ij} p^j$$

with $0 \le s_{ij} \le p-1$.

Proposition 3.1. *If $p > (3n-1)C_{\boldsymbol{\chi}} - n$ and $\boldsymbol{\chi}$ satisfies (1.2), then there is a unique m such that $v\big(\prod_{i=1}^n g(m+s_i)\big)$ is minimal.*

Proof. We may assume that $s_1 = q-1$ for simplicity. Write

$$m + s_i - (q-1)\epsilon_{i,-1} = \sum_{j=0}^{d-1} m_{ij} p^j,\ 1 \le i \le n$$

where $\epsilon_{i,-1} \in \{0,1\}$ is the integer part of $(m+s_i)/(q-1)$ and $0 \leq m_{ij} \leq p-1$. Then

$$m_{ij} = m_j + s_{ij} + \epsilon_{i,j-1} - p\epsilon_{ij}, \quad \epsilon_{ij} \in \{0,1\}, \epsilon_{i,-1} = \epsilon_{i,k-1}$$

and

$$v\Big(\prod_{i=1}^{n} g(m+s_i)\Big) = \sum_{i=1}^{n}\sum_{j=0}^{d-1} m_{ij} \tag{3.2}$$

by the Stickelberger's congruence theorem (3.1).

There exsits a permutation $\sigma_j \in S_n$ such that

$$s_{\sigma_j(1),j} \geq s_{\sigma_j(2),j} \geq \cdots \geq s_{\sigma_j(n),j}. \tag{3.3}$$

By Lemma 3.2, there exists a unique u_j such that

$$s_{\sigma_j(u_j),j} + \frac{p-1}{n}u_j = \max_{1\leq i\leq n}\left\{s_{\sigma_j(i),j} + \frac{p-1}{n}i\right\}.$$

Moreover,

$$s_{\sigma_j(u_j),j} + \frac{p-1}{n}u_j > s_{\sigma_j(i),j} + \frac{p-1}{n}i + 1 \tag{3.4}$$

for any $i \neq u_j$. Indeed, if $s_{\sigma_j(u_j)} \neq s_{\sigma_j(i)}$, then by Lemma 3.2, $s_{\sigma_j(u_j),j} \neq s_{\sigma_j(i),j}$ and this claim follows; if $s_{\sigma_j(u_j)} = s_{\sigma_j(i)}$, this follows from $(p-1)/n > 1$.

If $s_{ij} = s_{i'j}$, we have $\chi_i = \chi_{i'}$ and $\epsilon_{ij} = \epsilon_{i'j}$. If $s_{ij} > s_{i'j}$, then $s_{ij} + \epsilon_{i,j-1} \geq s_{i'j} + \epsilon_{i',j-1}$ and $\epsilon_{ij} \geq \epsilon_{i'j}$. By (3.3), there exists $0 \leq \mu_j \leq n$ such that

$$\epsilon_{\sigma_j(1),j} = \cdots = \epsilon_{\sigma_j(\mu_j),j} = 1, \quad \epsilon_{\sigma_j(\mu_j+1),j} = \cdots = \epsilon_{\sigma_j(n),j} = 0.$$

Notice that $s_1 = q-1, \epsilon_{1,k-1} = \epsilon_{1,-1} = 1$, which means $\mu_j \neq 0$. Since $\{s_{ij}+\epsilon_{i,j-1}\}_i$ has same order as (3.3), we have

$$m'_j := \min_i \{m_{ij}\} = m_j + s_{\sigma_j(\mu_j),j} + \epsilon_{\sigma_j(\mu_j),j-1} - p.$$

Then

$$\sum_i m_{ij} = \sum_i \big(m'_j + p(1-\epsilon_{ij}) + s_{ij} - s_{\sigma_j(\mu_j),j} + \epsilon_{i,j-1} - \epsilon_{\sigma_j(\mu_j),j-1}\big)$$

and the valuation (3.2) is

$$\begin{aligned}
&\sum_{i,j} m_{ij} = \sum_{i,j}\big(m'_j + p(1-\epsilon_{ij}) + s_{ij} - s_{\sigma_j(\mu_j),j} + \epsilon_{i,j-1} - \epsilon_{\sigma_j(\mu_j),j-1}\big)\\
&= \sum_j \Big(nm'_j + (p-1)(n-\mu_j) - ns_{\sigma_j(\mu_j),j} + \sum_i \big(s_{ij} + 1 - \epsilon_{ij} + \epsilon_{i,j-1} - \epsilon_{\sigma_j(\mu_j),j-1}\big)\Big)\\
&= \sum_j \Big(nm'_j + (p-1)(n-\mu_j) - ns_{\sigma_j(\mu_j),j} + \sum_i \big(s_{ij} + 1 - \epsilon_{\sigma_j(\mu_j),j-1}\big)\Big).
\end{aligned}$$

Write

$$E_{\sigma_j(1),j} = \cdots = E_{\sigma_j(u_j),j} = 1, \quad E_{\sigma_j(u_j+1),j} = \cdots = E_{\sigma_j(n),j} = 0.$$

If m is

$$M = \sum_{j=0}^{d-1} M_j p^j, \quad M_j = p - s_{\sigma_j(u_j),j} - E_{\sigma_j(u_j),j-1},$$

then $m'_{ij} = 0, \epsilon_{ij} = E_{ij}$ and $\mu_j = u_j$. Denote by V the corresponding valuation (3.2).

If all $\mu_j = u_j$, then $\epsilon_{ij} = E_{ij}$ and

$$\sum_{i,j} m_{ij} = V + n \sum_j m'_j \geq V.$$

If there exists j such that $\mu_j \neq u_j$, then

$$\begin{aligned}
&\sum_{i,j} m_{ij} - V \\
=&\sum_j \big(nm'_j + (p-1)(n-\mu_j) - ns_{\sigma_j(\mu_j),j} + \sum_i (s_{ij} + 1 - \epsilon_{\sigma_j(\mu_j),j-1})\big) \\
&\quad - \sum_j \big((p-1)(n-u_j) - ns_{\sigma_j(u_j),j} + \sum_i (s_{ij} + 1 - E_{\sigma_j(u_j),j-1})\big) \\
\geq& \sum_j \big((p-1)(u_j - \mu_j) + n(s_{\sigma_j(u_j),j} - s_{\sigma_j(\mu_j),j}) + \sum_i (E_{\sigma_j(u_j),j-1} - \epsilon_{\sigma_j(\mu_j),j-1})\big) \\
\geq& n \sum_j \big(s_{\sigma_j(u_j),j} + \frac{p-1}{n} u_j - s_{\sigma_j(\mu_j),j} - \frac{p-1}{n}\mu_j\big) + \sum_{i,j} (E_{\sigma_j(u_j),j} - \epsilon_{\sigma_j(\mu_j),j}) \\
=& n \sum_{\mu_j \neq u_j} \sum_j \big(s_{\sigma_j(u_j),j} + \frac{p-1}{n} u_j - s_{\sigma_j(\mu_j),j} - \frac{p-1}{n}\mu_j - 1\big) > 0
\end{aligned}$$

by (3.4). Hence the valuation (3.2) is minimal if and only if $m = M$. □

Lemma 3.2. *Assume that $p > (3n-1)C_{\boldsymbol{\chi}} - n$. If $\chi_i^n \neq \chi_{i'}^n$, then there is no integer $0 \leq \alpha \leq n$ such that $|s_{ij} - s_{i'j} - \frac{p-1}{n}\alpha| \leq 1$.*

Proof. There exists r, r' such that

$$s_i = \frac{(q-1)r}{C_{\boldsymbol{\chi}}}, \quad s_{i'} = \frac{(q-1)r'}{C_{\boldsymbol{\chi}}}$$

and

$$s_{ij} = \frac{a_{j+1}p - a_j}{C_{\boldsymbol{\chi}}}, \quad s_{i'j} = \frac{a'_{j+1}p - a'_j}{C_{\boldsymbol{\chi}}},$$

where $a_j \equiv rp^{-j}, a'_j \equiv r'p^{-j} \bmod C_{\boldsymbol{\chi}}$ with $1 \leq a_j, a'_j \leq C_{\boldsymbol{\chi}}$. Let $a''_j := a_j - a'_j$. Then $|a''_j| \leq C_{\boldsymbol{\chi}} - 1$.

If

$$\frac{p-1}{n}\alpha + t = s_{ij} - s_{i'j} = \frac{a''_{j+1}p - a''_j}{C_{\boldsymbol{\chi}}}$$

for some $0 \leq \alpha \leq n$ and $|t| \leq 1$, then

$$(na''_{j+1} - C_{\boldsymbol{\chi}}\alpha)p = na''_j - C_{\boldsymbol{\chi}}\alpha + nC_{\boldsymbol{\chi}}t.$$

There are three cases:

- If $na''_{j+1} - C_{\boldsymbol{\chi}}\alpha \neq 0$ and $\alpha = n$, then
  $$p \leq |(C_{\boldsymbol{\chi}} - a''_{j+1})p| = |C_{\boldsymbol{\chi}} - a''_j - C_{\boldsymbol{\chi}}t| \leq 3C_{\boldsymbol{\chi}} - 1 \leq (3n-1)C_{\boldsymbol{\chi}} - n$$
  since $n \geq 2$.
- If $na''_{j+1} - C_{\boldsymbol{\chi}}\alpha \neq 0$ and $\alpha < n$, then

$$p \leq \big|na''_j - C_{\boldsymbol{\chi}}\alpha + nC_{\boldsymbol{\chi}}t\big| \leq n(C_{\boldsymbol{\chi}} - 1) + C_{\boldsymbol{\chi}}(n-1) + nC_{\boldsymbol{\chi}} \leq (3n-1)C_{\boldsymbol{\chi}} - n.$$

- If $na''_{j+1} - C_{\boldsymbol{\chi}}\alpha = 0$, both of $na''_j = C_{\boldsymbol{\chi}}(\alpha - nt)$ and $na''_{j+1} = C_{\boldsymbol{\chi}}\alpha$ are multipliers of $C_{\boldsymbol{\chi}}$ since $nt \in \mathbb{Z}$. That's to say, $(\chi_i\chi_{i'}^{-1})^n$ is trivial and then $\chi_i^n = \chi_{i'}^n$.

The result then follows. □

Remark 3.3. When $n = 2$, $p > 3C_{\boldsymbol{\chi}} - 2$ is enough by a careful estimate, see [Zha21, Lemma 3.4, Proposition 3.6].

4. The generating fields

4.1. **The proof.**

Proof of Theorem 1.4. Note that if $\boldsymbol{\chi}$ is Kummer-induced, there is a non-trivial character Λ such that $\boldsymbol{\chi} = \boldsymbol{\chi}\Lambda$ and $\Lambda^n = 1$. Thus there exists $i \neq j$ such that $\chi_i = \chi_j\Lambda$ and $\chi_i^n = \chi_j^n$, which contradicts to our assumptions. Certainly, $\boldsymbol{\chi} = (\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$ is also impossible.

By Theorem 1.2, 1.3, the fact

$$\sigma_t\tau_w \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \prod \boldsymbol{\chi}^{-w}(t)\mathrm{Kl}_n(\psi, \boldsymbol{\chi}^w, q, at^n),$$

and $t^p = t$, we have

$$t^n = a^{1-p^k}, \ \boldsymbol{\chi}^w = \eta\boldsymbol{\chi}^{p^k}, \ \eta(a) = \prod \boldsymbol{\chi}^w(t)$$

for some integer k.

Recall that $n_1 = (n, p-1)$. Denote by $b = a^{(1-p)/n_1}$. Then $q_1 = p^{d_1}$ where d_1 is the degree of b. Write $n = n_1n_2$ and $n_0 \equiv n_2^{-1} \bmod (p-1)$. Then

$$t^{n_1} = t^{nn_0} = a^{n_0(1-p^k)} = b^{n_0n_1(p^k-1)/(p-1)}$$

and

$$1 = t^{p-1} = b^{n_0(p^k-1)}.$$

Since the degree of b^{n_0} is d_1, the same as the degree of b, we have $k = d_1\beta$ for some integer β. Conversely, if $k = d_1\beta$ and $b^{n_0(p^k-1)} = 1$, then

$$t^{n_1} = b^{n_0n_1(p^k-1)/(p-1)} = a_1^{n_1(p^k-1)/(p^{d_1}-1)} = a_1^{n_1\beta}$$

has solutions $t = \lambda a_1^\beta$ for some $\lambda^{n_1} = 1$. □

Recall $c \mid (q-1)$ is the least common multiplier of orders of χ_i. By abuse of notations, we also denote by $\tau_w \in \mathrm{Gal}\big(\mathbb{Q}(\mu_{pc})/\mathbb{Q}\big)$ for $w \in (\mathbb{Z}/c\mathbb{Z})^\times$ similarly.

Remark 4.1. Fix q, $\boldsymbol{\chi}$, a and assume that $\boldsymbol{\chi}$ satisfies (1.2). Consider the Kloosterman sums

$$S_k = \mathrm{Kl}(\psi, \boldsymbol{\chi} \circ \mathbf{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}, q^k, a).$$

By Theorem 1.4, if $p > \max\left\{(2n^{2dk}+1)^2, (3n-1)C_{\boldsymbol{\chi}} - n\right\}$, then $\mathbb{Q}(S_k) = \mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t\tau_w$ such that there exists an integer β and a character η on $\mathbb{F}_q^\times$ satisfying

$$t = \lambda a_1^\beta, \lambda^{n_1} = 1, \quad \boldsymbol{\chi}^w = \eta\boldsymbol{\chi}^{q_1^\beta}, \quad \eta(a) = \gamma \cdot \prod \boldsymbol{\chi}^w(t), \gamma^k = 1. \tag{4.1}$$

Thus $\mathbb{Q}(S_k) = \mathbb{Q}(S_{k-c})$ since $\gamma^c = 1$. The L-function

$$L(T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} S_k\right)$$

is a rational function over $\mathbb{Q}(\zeta_{p(q-1)})$ by the Dwork-Bombieri-Grothendick rationality theorem. Thus the sequence $\{S_k\}_k$ is linear recurrence sequence. As shown in [WY20, Theorem 3], the sequence $\{\mathbb{Q}(S_k)\}_{k\geq N}$ is periodic of period r for some N. Thus if $p > \max\left\{\left(2n^{2d(N+r)}+1\right)^2, (3n-1)C_{\boldsymbol{\chi}} - n\right\}$, the generating field of S_k is determined by (4.1) for any k. For this purpose, we need to decrease the bound $(2n^{2d}+1)^2$ and estimate the period r and N. We guess that S_k has the predicted generating field if $p > 3ndc$.

We will end our paper with two examples.

4.2. **An example: $n = 2$ case.**

Proposition 4.2. *Let $\boldsymbol{\chi} = \{1, \chi\}$, where χ is a multiplicative character of order $c \neq 2$. If $p > \max\{(2^{2d+1}+1)^2, 5c-2)\}$, then $\mathrm{Kl}(\psi, \boldsymbol{\chi}, p^d, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where*

$$H = \begin{cases} \langle \tau_{q_1}\sigma_{a_1}, \sigma_{-1}, \tau_{-1}\rangle, & \text{if } \chi(-1) = 1, \chi(a) = 1; \\ \langle \tau_{-q_1}\sigma_{a_1}, \sigma_{-1}\rangle, & \text{if } \chi(-1) = 1, \chi(a) = \chi(a_1) = -1; \\ \langle \tau_{q_1^\alpha}\sigma_{a_1^\alpha}, \sigma_{-1}\rangle, & \text{if } \chi(-1) = 1, \chi(a)^\alpha \neq 1; \\ \langle \tau_{q_1}\sigma_{-a_1}, \tau_{-1}\sigma_{-1}\rangle, & \text{if } \chi(-1) = -1, \chi(a) = \chi(a_1) = -1; \\ \langle \tau_{q_1}\sigma_{a_1}, \tau_{-1}\rangle & \text{if } \chi(-1) = -1, \chi(a) = 1; \\ \langle \tau_{q_1}\sigma_{a_1}, \tau_{-1}\sigma_{-1}\rangle, & \text{if } \chi(-1) = -1, \chi(a) = -1, \chi(a_1) = 1; \\ \langle \tau_{q_1^{\alpha/2}}\sigma_{-a_1^{\alpha/2}}\rangle, & \text{if } \chi(-1) = -1, 2 \mid \alpha, \chi(a) \neq \pm 1; \\ \langle \tau_{q_1^\alpha}\sigma_{a_1^\alpha}\rangle, & \text{if } \chi(-1) = -1, 2 \nmid \alpha, \chi(a) \neq \pm 1. \end{cases}$$

is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\mu_{pc})/\mathbb{Q})$, $q_1 = \#\mathbb{F}_p(a^{(1-p)/2}), a_1 = a^{(1-q_1)/2}$ and α is the order of $\chi(a_1) \in \mu_{p-1}$.

Proof. We have

$$\boldsymbol{\chi}^w = \{1, \chi^w\} = \eta\boldsymbol{\chi}^{q_1^\beta} = \left\{\eta, \eta\chi^{q_1^\beta}\right\}.$$

There are two cases:
(a) If $\eta = 1, \chi^w = \chi^{q_1^\beta}$, then $w \equiv q_1^\beta \bmod c$. Since $\eta(a) = \chi^w(t)$, we have $1 = \chi(t) = \chi(\pm a_1^\beta)$.
(b) If $\eta = \chi^w, \eta\chi^{q_1^\beta} = 1$, then $w \equiv -q_1^\beta \bmod c$. Since $\eta(a) = \chi^w(t) = \chi(t)^{-1}$ and $t = \pm a_1^\beta$, we have $\chi(a) = \chi(t) = \chi(\pm a_1^\beta)$. Since $a_1 = a^{(1-q_1)/2} \in \mathbb{F}_p^\times$, we have

$$\chi(a_1)^2 = \chi(a)^{1-q_1} = \chi(a_1)^{(1-q_1)\beta} = 1.$$

Thus $\chi(a_1) = \pm 1$ and $\alpha = 1$ or 2.

Case $\chi(-1) = 1$: In case (a), $\beta = \alpha m$ for some m and $w \equiv q_1^{\alpha m}, t = \pm a_1^{\alpha m}$. In case (b), if $\alpha = 1$, $\chi(a_1) = \chi(a) = 1$, then $w \equiv -q_1^m, t = \pm a_1^m$; if $\alpha = 2$, $\chi(a_1) = \chi(a) = -1$, then $w \equiv -q_1^{1+2m}, t = \pm a_1^{1+2m}$.

Case $\chi(-1) = -1$ and $2 \mid \alpha$: In case (a), $w \equiv q_1^{\alpha m}, t = a_1^{\alpha m}$ or $w \equiv q_1^{\alpha(m+1/2)}, t = -a_1^{\alpha(m+1/2)}$. In case (b), $\alpha = 2$, $\chi(a) = \chi(a_1) = -1$. Then $w \equiv -q_1^{1+2m}, t = a_1^{1+2m}$ or $w \equiv -q_1^{2m}, t = -a_1^{2m}$.

Case $\chi(-1) = -1$ and $2 \nmid \alpha$: In case (a), $w \equiv q_1^{\alpha m}, t = a_1^{\alpha m}$. In case (b), $\alpha = 1$ and $\chi(a_1) = 1$. If $\chi(a) = 1$, then $w \equiv -q_1^m, t = a_1^m$; if $\chi(a) = -1$, then $w \equiv -q_1^m, t = -a_1^m$. ☐

Example 4.3. If $a \in \mathbb{F}_p^\times$, then $q_1 = p, \alpha = 1$ or 2. One can easily obtain that

$$H = \begin{cases} \langle \tau_p, \tau_{-1}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1 \text{ and } \chi(a) = 1; \\ \langle \tau_p, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1 \text{ and } \chi(a) = -1; \\ \langle \tau_p, \tau_{-1} \rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) = 1; \\ \langle \tau_p, \tau_{-1}\sigma_{-1} \rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) = -1; \\ \langle \tau_p \rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) \neq \pm 1. \end{cases}$$

This drops the combinatorial condition on (p, d) and the non-vanishing condition on $\mathrm{Tr}(a)$ in [Zha21, Theorems 1.1, 1.3], while we require that p is large with respect to d.

Remark 4.4. When $\chi = \Lambda_2$, $\Lambda_2(a) = 1$, if we assume that $\mathrm{Tr}(\sqrt{a}) \neq 0$, then the Kloosterman sum generates $\mathbb{Q}(\mu_p)^+$ if $\chi(-1) = 1$; $\mathbb{Q}(\mu_p)$ if $\chi(-1) = -1$. See [Zha21, Theorem 1.1(1)].

4.3. **An example with trivial η.**

Example 4.5. Let $\boldsymbol{\chi}$ be a n-tuple containing **1** and satisfying (1.2). Assume that χ_i, χ_j have same multiplicities only if $\chi_i = \chi_j$. It's easy to see that $\eta = 1$ and $\chi_i^w = \chi_i^{q_1^\beta}$. Thus $w \equiv q_1^\beta \bmod c$. Write $\chi = \prod \boldsymbol{\chi}$ and denote by ℓ the minimal positive integer such that

$$\chi(a_1)^\beta \in \{\chi(\lambda) \mid \lambda^{n_1} = 1\}.$$

Write $\chi(a_1)^\ell = \chi(\lambda_0^{-1})$ and $t_0 = \lambda_0 a_1^\ell$. If $t = \lambda_1 a_1^\beta$ for some $\lambda_1^{n_1} = 1$ with $\chi(t) = 1$, then $\beta = \ell m$ and $t = \lambda t_0^m$ for some $\lambda^{n_1} = 1$ and $\chi(\lambda) = 1$. Hence if $p > \max\{(2n^{2d}+1)^2, (3n-1)C_{\boldsymbol{\chi}} - n\}$, then $\mathrm{Kl}(\psi, \boldsymbol{\chi}, p^d, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where

$$H = \langle \tau_{q_1^\ell}\sigma_{t_0}, \sigma_\lambda \mid \lambda^{n_1} = 1, \prod \boldsymbol{\chi}(\lambda) = 1 \rangle.$$

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School of Mathematics, Hefei University of Technology, Hefei, Anhui 230009, China
Email address: `zsxqq@mail.ustc.edu.cn`