ON LINEARITY OF THE PERIODS OF SUBTRACTION GAMES

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ABSTRACT. The subtraction game is an impartial combinatorial games involving a finite set S of positive integers. The nim-sequence $\mathcal{G}_S$ associated to this game is ultimately periodic. In this paper, we study the nim-sequence $\mathcal{G}_{S\cup\{c\}}$ where S is fixed and c varies. We conjecture that there is a multiplier q of the period of $\mathcal{G}_S$, such that for sufficiently large c, the pre-period and period of $\mathcal{G}_{S\cup\{c\}}$ are linear on c, if c modulo q is fixed. We prove it in several cases.

We also give new examples with period 2 inspired by this conjecture.

CONTENTS

1. INTRODUCTION

Let S be a finite set of positive integers. The *(finite) subtraction game* $\text{SUB}(S)$ is a two-player game involving a heap of $n \geq 0$ counters. The two players move alternately, subtracting some $s \in S$ counters. The player who cannot make a move loses.

We always write the subtraction set as $S = \{s_1, \ldots, s_k\}$ with an order $s_1 < s_2 < \cdots < s_k$. Denote by $\mathcal{G}(n) = \mathcal{G}_S(n)$ the *nim-value* (or *Grundy-value*), i.e.,

$$\mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-s) : s \in S, s \leq n\}, \quad \forall n \geq 0,$$

where mex means the minimal non-negative integer not in the set. The sequence $\mathcal{G} = \mathcal{G}_S = \{\mathcal{G}(n)\}_{n\geq 0}$ is called the *nim-sequence*.

If $d = \gcd(S) = \gcd\{s : s \in S\} > 1$ and $S' = \{s/d : s \in S\}$, then $\mathcal{G}_S(n) = \mathcal{G}_{S'}(m)$, where $md \leq n < (m+1)d$. Hence we may assume that $\gcd(S) = 1$ if necessary.

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Definition 1.1. A subtraction game $\textsc{sub}(S)$ (or its nim-sequence $\mathcal{G}$) is called *ultimately periodic*, if there exist integers $p \geq 1$ and $\ell \geq 0$ such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq \ell$. The minimal p is called the *period* and the minimal ℓ is called the *pre-period*.

Since $\mathcal{G}(n) \leq k$, one can show that $\mathcal{G}$ is ultimately periodic by the pigeonhole principle, see [ANW07, Theorem 7.22]. We have the following lemma to determine the period and pre-period.

Lemma 1.2 ([ANW07, Corollary 7.34])**.** *The minimal integers $\ell \geq 0, p \geq 1$ such that $\mathcal{G}(n) = \mathcal{G}(n+p)$ for $\ell \leq n < \ell + s_k$ are the pre-periodic and period of $\mathcal{G}$ respectively.*

In this paper, we will propose a conjecture (Conjecture 5.5) on $\textsc{sub}(S \cup \{c\})$ where S is fixed and c varies. More precisely, there is a positive integer q which is a multiplier of the period of $\textsc{sub}(S)$, such that for each $0 \leq r < q-1$, the pre-period and period of $\textsc{sub}(S \cup \{c\})$ are linear on $c = qt + r$, while t is large enough. We will prove it in several cases. We also give new nim-sequences with period 2 inspired by this conjecture.

Let t, a be a non-negative integer and $\mathcal{H} = (h_1 \cdots h_k)$ a sequence of integers with finite length. As usual, we denote by a^t the sequence $\underbrace{a \cdots a}_{t \text{ copies}}$ and $\mathcal{H}^t$ the sequence $\underbrace{\mathcal{H} \cdots \mathcal{H}}_{t \text{ copies}}$. Denote by $\underline{\mathcal{H}}$ the infinite-length sequence with periodic sequence $\mathcal{H}$, i.e., $\underline{\mathcal{H}} = \mathcal{H}\mathcal{H}\cdots$. For example, if ℓ and p is the pre-period and period of a nim-sequence $\mathcal{G}$ respectively, then we can write

$$\mathcal{G} = \mathcal{G}(0)\mathcal{G}(1)\mathcal{G}(2)\cdots = \mathcal{G}(0)\cdots\mathcal{G}(\ell-1)\underline{\mathcal{G}(\ell)\cdots\mathcal{G}(\ell+p-1)}.$$

We will not give the detailed proof of each nim-sequence, since the proof is by a lengthy and tedious induction.

2. The case $S = \{1, b, c\}$

In this section, we will consider nim-sequence when $S = \{1, b, c\}$. Let's recall some classical cases firstly.

Lemma 2.1. *Denote by p the period of $\textsc{sub}(S)$. If the pre-period of $\textsc{sub}(S)$ is zero, then $\mathcal{G}_{S \cup \{x+pt\}} = \mathcal{G}_S$ for any $x \in S$ and $t \geq 1$.*

Proof. Certainly $\mathcal{G}_{S'}(0) = \mathcal{G}_S(0) = 0$ where $S' = S \cup \{x + pt\}$. Suppose that $\mathcal{G}_{S'}(i) = \mathcal{G}_S(i)$ for $0 \leq i \leq n-1$. If $n < x + pt$, then

$$\mathcal{G}_{S'}(n) = \operatorname{mex}\{\mathcal{G}(n-s) : s \in S, s \leq n\} = \mathcal{G}(n).$$

If $n \geq x + pt$, then

$$\begin{aligned}\mathcal{G}_{S'}(n) &= \operatorname{mex}\{\mathcal{G}(n-x-pt), \mathcal{G}(n-s) : s \in S, s \leq n\} \\ &= \operatorname{mex}\{\mathcal{G}(n-x), \mathcal{G}(n-s) : s \in S, s \leq n\} = \mathcal{G}(n).\end{aligned}$$

The lemma then follows by induction. □

Example 2.2. The nim-sequence of $\textsc{sub}(1)$ is $\underline{01}$. If $1 \in S$ and the elements of S are all odd, then the nim-sequence $\mathcal{G}_S = \underline{01}$ by applying Lemma 2.1 several times. In fact, this condition is also necessary, see [CH10].

Example 2.3. Let $S = \{a, c\}$ with $a > 1$. Write $c = at + r, 0 \le r < a$. Then

$$\mathcal{G} = \begin{cases} \underline{(0^a 1^a)^{t/2} 0^r 2^{a-r} 1^r}, & t \text{ is even;} \\ \underline{(0^a 1^a)^{(t+1)/2} 2^r}, & t \text{ is odd,} \end{cases}$$

$\ell = 0$ and $p = c + a$ or $2a$. See [BCG03].

Example 2.4. Let $S = \{1, b, c\}$ with odd b. Then

$$\mathcal{G} = \underline{(01)^{c/2}(23)^{(b-1)/2}2},$$

$\ell = 0$ and $p = c + b$.

Example 2.5. Let $S = \{1, 2, c\}$. Note that $\mathcal{G}_{\{1,2\}} = \underline{012}$ with period 3. Write $c = 3t + r, 0 \le r < 3$.

(1) If $r = 1, 2$, then $\mathcal{G} = \mathcal{G}_{\{1,2\}}$, $\ell = 0$ and $p = 3$ by Lemma 2.1.
(2) If $r = 0$, then $\mathcal{G} = \underline{(012)^t 3}$, $\ell = 0$ and $p = c + 1$.

Example 2.6. Let $S = \{1, 4, c\}$. Denote by $\mathcal{H} = 01012$, then $\mathcal{G}_{\{1,4\}} = \underline{\mathcal{H}}$ with period 5. Write $c = 5t + r, 0 \le r < 5$.

(1) If $r = 1, 4$, then $\mathcal{G} = \mathcal{G}_{\{1,4\}}$, $\ell = 0$ and $p = 5$ by Lemma 2.1.
(2) If $r = 2$, then $\mathcal{G} = \underline{\mathcal{H}^t\, 012}$, $\ell = 0$ and $p = c + 1$.
(3) If $r = 3$, then $\mathcal{G} = \underline{\mathcal{H}^{t+1}\, 32}$, $\ell = 0$ and $p = c + 4$.
(4) If $r = 0, c = 5$, then $\mathcal{G} = \underline{\mathcal{H}\, 323}$, $\ell = 0$ and $p = 8$.
(5) If $r = 0, c > 5$, then $\mathcal{G} = \mathcal{H}^t\, 323013\underline{\mathcal{H}^{t-1}012012}$, $\ell = c + 6$ and $p = c + 1$.

Theorem 2.7. *Let $S = \{1, b, c\}$, where $b = 2k \ge 6$ is even. Write $c = t(b + 1) + r$ with $0 \le r \le b$.*

(1) *If $r = 1, b$, then $\ell = 0$ and $p = b + 1$.*
(2) *If $3 \le r \le b - 1$ is odd, then $\ell = 0$ and $p = c + b$.*
(3) *If $r = b - 2$, then $\ell = 0$ and $p = c + 1$.*
(4) *If $c = b + 1$, then $\ell = 0, p = 2b = c + b - 1$;*
(5) *If $c > b + 1$, $0 \le r \le b - 4$ is even and $t + r/2 \ge k$, then $\ell = \left(\frac{b-r}{2} - 1\right)(c + b + 2) - b$ and $p = c + 1$.*
(6) *If $c > b+1$, $0 \le r \le b-4$ is even and $t+r/2 \le k-1$, then $\ell = t(c+b+2)-b$. If $t + r/2 < k - 1$, then $p = c + b$; if $t + r/2 = k - 1$, then $p = b - 1$.*

Proof. Denote by $\mathcal{H} = (01)^k 2$, then $\mathcal{G}_{\{1,b\}} = \underline{\mathcal{H}}$ with period $b + 1$.
(1) In this case, $\mathcal{G} = \mathcal{G}_{\{1,b\}}$, $\ell = 0$ and $p = b + 1$ by Lemma 2.1.
(2) In this case, $\mathcal{G} = \underline{\mathcal{H}^{t+1}\, (32)^{(r-1)/2}}$, $\ell = 0$ and $p = c + b$.
(3) In this case, $\mathcal{G} = \underline{\mathcal{H}^t\, (01)^{k-1} 2}$, $\ell = 0$ and $p = c + 1$.
(4) In this case, $\mathcal{G} = \underline{(01)^k (23)^k} = \underline{\mathcal{H}3(23)^{k-1}}$, $\ell = 0$ and $p = 2b = b + c - 1$.

(5) Write $r=2v$. When $1\le v\le k-2$, the first $(c+1)(k-v+1)$ terms of $\mathcal{G}$ are (the bold part is the first periodic nim-sequence)

i	$\mathcal{G}\big((c+1)i+j\big),\ 0\le j\le c$
0	$\mathcal{H}^t, (01)^v2$
1	$(32)^{k-v-1}(01)^{v+1}2, \mathcal{H}^{t-1}, (01)^v0$
2	$1(01)^{k-v-2}2(01)^{v+1}2, (32)^{k-v-2}(01)^{v+2}2, \mathcal{H}^{t-2}, (01)^v0$
i	$1(01)^{k-v-2}2(01)^{v+1}0,\dots,1(01)^{k-v-i+1}2(01)^{v+i-2}0,$ $1(01)^{k-v-i}2(01)^{v+i-1}2, (32)^{k-v-i}(01)^{v+i}2, \mathcal{H}^{t-i}, (01)^v0$
$k-v-1$	$1(01)^{k-v-2}2(01)^{v+1}0,\dots,1(01)^22(01)^{k-3}0, 1(01)2(01)^{k-2}2,$ $(3\mathbf{2})^{\mathbf{1}}(\mathbf{01})^{\mathbf{k-1}}\mathbf{2}, \mathcal{H}^{\mathbf{t-k+v+1}}, (\mathbf{01})^{\mathbf{v}}\mathbf{0}$
$k-v$	$\mathbf{1(01)^{k-v-2}2(01)^{v+1}0},\dots,\mathbf{1(01)2(01)^{k-2}0}, \mathbf{1}2(01)^{k-1}2,$ $\mathcal{H}^{t-k+v-1}, (01)^v0.$

When $v=0$, the first $(c+1)(k+1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big),\ 0\le j\le c$
0	$\mathcal{H}^t3$
1	$(23)^{k-1}013, \mathcal{H}^{t-1}0$
2	$1(01)^{k-2}2(01)2, (32)^{k-2}(01)^22, \mathcal{H}^{t-2}0$
i	$1(01)^{k-2}2(01)0,\cdots,1(01)^{k-i+1}2(01)^{i-2}0, 1(01)^{k-i}2(01)^{i-1}2,$ $(32)^{k-i}(01)^i2, \mathcal{H}^{t-i}0$
$k-1$	$1(01)^{k-2}2(01)0,\cdots,1(01)^22(01)^{k-3}0, 1(01)^12(01)^{k-2}2,$ $(3\mathbf{2})^{\mathbf{1}}(\mathbf{01})^{\mathbf{k-1}}\mathbf{2}, \mathcal{H}^{\mathbf{t-k+1}}\mathbf{0}$
k	$\mathbf{1(01)^{k-2}2(01)0},\cdots,\mathbf{1(01)^12(01)^{k-2}0}, \mathbf{1}2(01)^{k-1}2, \mathcal{H}^{t-k+1}0.$

In both cases, we have $\ell=\left(\frac{b-r}{2}-1\right)(c+b+2)-b$, $p=c+1$ and

$$\mathcal{G}=\cdots\underline{2(01)^{k-1}\big(2(01)^k\big)^{t-k+v+1}\big(2(01)^{k-1}\big)^{k-v-1}}.$$

(6) When $1\le v\le k-2$, the first $(c+1)(t+2)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big), 0\le j\le c$
0	$\mathcal{H}^t\,(01)^v2$
1	$(32)^{k-v-1}(01)^{v+1}2, \mathcal{H}^{t-1}(01)^v0$
2	$1(01)^{k-v-2}2(01)^{v+1}2, (32)^{k-v-2}(01)^{v+2}2, \mathcal{H}^{t-2}(01)^v0$
i	$1(01)^{k-v-2}2(01)^{v+1}0,\dots,1(01)^{k-v-i+1}2(01)^{v+i-2}0,$ $1(01)^{k-v-i}2(01)^{v+i-1}2, (32)^{k-v-i}(01)^{v+i}2, \mathcal{H}^{t-i}(01)^v0$
$t-1$	$1(01)^{k-v-2}2(01)^{v+1}0,\dots,1(01)^{k-v-t+2}2(01)^{v+t-3}0,$ $1(01)^{k-v-t+1}2(01)^{v+t-2}2, (32)^{k-v-t+1}(01)^{v+t-1}2, \mathcal{H}^1(01)^v0$
t	$1(01)^{k-v-2}2(01)^{v+1}0,\dots,1(01)^{k-v-t+1}2(01)^{v+t-2}0,$ $1(01)^{k-v-t}2(01)^{v+t-1}2, (3\mathbf{2})^{\mathbf{k-v-t}}(\mathbf{01})^{\mathbf{v+t}}\mathbf{2}, (\mathbf{01})^{\mathbf{v}}\mathbf{0}$
$t+1$	$\mathbf{1(01)^{k-v-2}2(01)^{v+1}0},\dots,\mathbf{1(01)^{k-v-t+1}2(01)^{v+t-2}0},$ $\mathbf{1(01)^{k-v-t}2(01)^{v+t-1}0}, \mathbf{1(01)^{k-v-t-1}2(01)^{v+t}}2, (32)^{k-v-t-1}01\cdots$

We have $\ell=t(c+b+2)-b$. If $t+v<k-1$, we have $p=c+b$ and

$$\mathcal{G}=\cdots\underline{2(32)^{k-v-t-1}(01)^{v+t}2[(01)^{k-1}2]^t(01)^{v+t}}.$$

If $t+v=k-1$, we have $p=b-1$ and $\mathcal{G}=\cdots\underline{2(01)^{k-1}}$.

When $v = 0$, the first $(c+1)(t+2)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big), 0 \le j \le c$
0	$\mathcal{H}^t\,3$
1	$(23)^{k-1}013, \mathcal{H}^{t-1}0$
2	$1(01)^{k-2}2(01)2, (32)^{k-2}(01)^2 2, \mathcal{H}^{t-2}0$
i	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-i+1}2(01)^{i-2}0, 1(01)^{k-i}2(01)^{i-1}2,$ $(32)^{k-i}(01)^i 2, \mathcal{H}^{t-i}0$
$t-1$	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-t+2}2(01)^{t-3}0, 1(01)^{k-t+1}2(01)^{t-2}2,$ $(32)^{k-t+1}(01)^{t-1}2, \mathcal{H}^1 0$
t	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-t+1}2(01)^{t-2}0, 1(01)^{k-t}2(01)^{t-1}2,$ $(3\mathbf{2})^{\mathbf{k-t}}\mathbf{(01)^t 20}$
$t+1$	$\mathbf{1(01)^{k-2}2(01)0}, \cdots, \mathbf{1(01)^{k-t}2(01)^{t-1}0}, \mathbf{1(01)^{k-t-1}2(01)^t 0},$ $\mathbf{1(01)^{k-t-1}2(01)^t}2, (32)^{k-t-1}01\cdots$

We have $\ell = t(c+b+2) - b$. If $t < k-1$, we have $p = c+b$ and

$$\mathcal{G} = \cdots \underline{2(32)^{k-t-1}(01)^t 2[(01)^{k-1}2]^t(01)^t}.$$

If $t = k-1$, we have $p = b-1$ and $\mathcal{G} = \cdots \underline{2(01)^{k-1}}$. □

Remark 2.8. The case $c < 4b$ is studied in [Ho15], but there are some incorrect data. In Table 1, $p = a-1$ if $r = a-3 \ge 3$. In Table B.11, $n_0 = a+2b+4$ if $2 \le r \le a-4$. In Table B.12, $n_0 = 2a+3b+6$ if $3 \le r \le a-5$. The corresponding pre-period nim-values also need to be modified.

3. The case $S = \{a, 2a, c\}$

Theorem 3.1. *Let $S = \{a, 2a, c\}$. Write $c = 3ta + r$ with $0 \le r < 3a$. Then*

$$\ell = \begin{cases} 3ta + a = c + a - r, & 0 < r < a; \\ 0, & \textit{otherwise}. \end{cases}, \quad p = \begin{cases} 3a/2, & r = a/2; \\ 3a, & a/2 < r \le 2a; \\ c + a, & \textit{otherwise}. \end{cases}$$

Proof. Denote by $\mathcal{H} = 0^a 1^a 2^a$, then $\mathcal{G}_{\{a,2a\}} = \underline{\mathcal{H}}$ with period $q = 3a$. Write $a = 2k-1$ if a is odd; $a = 2k$ if a is even.

(1) If $a \le r \le 2a$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = 3a$.

(2) If $r = 0$, then $\mathcal{G} = \underline{\mathcal{H}^t 3^a}$, $\ell = 0$ and $p = c + a$.

(3) If $0 < r < k$, then

$$\mathcal{G} = \mathcal{H}^t 0^r 3^{a-r} \underline{(1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r})^t 1^r 0^r 3^{a-2r} 2^r},$$

$\ell = 3at + a$ and $p = c + a$.

(4) If $k \le r < a$, then

$$\mathcal{G} = \mathcal{H}^t 0^r 3^{a-r} \underline{1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r}},$$

$\ell = 3at + a$ and $p = 3a$ or $3a/2$.

(5) If $r > 2a$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} 3^{r-2a}},$$

$\ell = 0$ and $p = c + a$. □

4. The case S contains successive numbers

Theorem 4.1. *Let* $S = \{a, a+1, \dots, b-1, b, c\}$. *Write* $c = t(a+b) + r$ *with* $0 \le r < a+b$. *Then*

$$\ell = 0, \quad p = \begin{cases} a+b, & a \le r \le b; \\ c+a, & r = 0 \text{ or } r > b; \\ c+b, & 0 < r < a. \end{cases}$$

Proof. Write $b = ak + s$, $0 \le s \le a-1$ and denote by $\mathcal{H} = 0^a 1^a \cdots k^a (k+1)^s$, then $\mathcal{G}_{\{a,a+1,\dots,b\}} = \underline{\mathcal{H}}$ with period $q = a + b = a(k+1) + s$.

(1) If $a \le r \le b$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = a+b$ by Lemma 2.1.

(2) If $r = 0$, then

$$\mathcal{G} = \underline{\mathcal{H}^t (k+1)^{a-s} (k+2)^s}.$$

If $r > b$ and $r + s > q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k+1)^{a-s} (k+2)^{r+s-q}}.$$

If $r > b$ and $r + s \le q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k+1)^{a+r-q}}.$$

In all cases, we have $\ell = 0$ and $p = c + a$.

(3) If $0 < r < a - 2s$, then

$$\begin{aligned} \mathcal{G} =& \underline{\mathcal{H}^t, 0^r (k+1)^{a-s-r} (k+2)^s, 1^r (k+2)^{a-s-r} (k+3)^s, \cdots,} \\ & \underline{(k-1)^r (2k)^{a-s-r} (2k+1)^s, k^r (2k+1)^s}. \end{aligned}$$

If $a - 2s \le r < a - s$, then

$$\begin{aligned} \mathcal{G} =& \underline{\mathcal{H}^t, 0^r (k+1)^{a-s-r} (k+2)^s, 1^r (k+2)^{a-s-r} (k+3)^s, \cdots,} \\ & \underline{(k-1)^r (2k)^{a-s-r} (2k+1)^s, k^r (2k+1)^{a-s-r} (2k+2)^{2s+r-a}}. \end{aligned}$$

If $a - s \le r < a$, then

$$\begin{aligned} \mathcal{G} =& \underline{\mathcal{H}^t, 0^r (k+2)^{a-r}, 1^r (k+3)^{a-r}, \cdots,} \\ & \underline{(k-1)^r (2k+1)^{a-r}, k^r (k+1)^s}, \end{aligned}$$

In all cases, we have $\ell = 0$ and $p = c + b$. □

5. Linearity on pre-periods and periods

Let S be a fixed subtraction set. We denote by ℓ_p the pre-period and p_c the period of $\text{SUB}(S \cup \{c\})$.

Example 5.1. Let $S = \{6, 17\}$. Then $\mathcal{G} = \underline{0^6 1^6 0^5 2 1^5}$ with period 23. Write $c = 23t + r, 0 \le r < 23$. For $116 \le c \le 500$, we have

$$\ell_c = \begin{cases} 9c + 147, & r = 0, 12; \\ 7c + 112, & r = 1, 13; \\ 5c + 77, & r = 2, 14; \\ 3c + 42, & r = 3, 15; \\ c + 7, & r = 4, 16; \\ 0, & \text{otherwise}, \end{cases} \qquad p_c = \begin{cases} c + 6, & 0 \le r \le 5 \text{ or } 12 \le r \le 16; \\ c + 17, & 7 \le r \le 11 \text{ or } 18 \le r \le 22; \\ 23, & \text{otherwise}. \end{cases}$$

Example 5.2. Let $S = \{3, 5, 8\}$. Then $\mathcal{G} = \underline{0^3 1^3 2^3 3^2}$ with period 11. Write $c = 11t + r, 0 \le r < 11$. For $c \le 500$, we have

$$\ell_c = \begin{cases} d + 18, & r = 1, 2; \\ 0, & \text{otherwise,} \end{cases} \qquad p_c = \begin{cases} d + 3, & r = 0, 1, 9, 10; \\ d + 25, & r = 2; \\ 11, & \text{otherwise.} \end{cases}$$

Example 5.3. Let $S = \{2, 3, 5, 7\}$. Then $\mathcal{G} = \underline{0^2 1^2 2^2 3^2 4}$ with period 9. Write $c = 18t + r, 0 \le r < 18$. For $c \le 500$, we have

$$\ell_c = \begin{cases} 2d - 4, & r = 1; \\ d + 5, & r = 10; \\ 0, & \text{otherwise,} \end{cases} \qquad p_c = \begin{cases} d + 2, & r = 0, 8, 9, 10, 17; \\ 4, & r = 1; \\ 9, & \text{otherwise.} \end{cases}$$

Example 5.4. Let $S = \{4, 11, 12, 14\}$. Then $\mathcal{G} = \cdots \underline{20^4 1^4 0^3 31^3 2^3 03^3 12}$ with pre-period 24 and period 25. Write $c = 25t + r, 0 \le r < 25$. For $101 \le c \le 500$, we have

$$\ell_c = \begin{cases} 4c + 91, & r = 0; & 2c + 34, & r = 2; & c + 14, & r = 19; \\ 3c + 4, & r = 6; & 2c + 36, & r = 5; & c + 26, & r = 9; \\ 3c + 5, & r = 22; & 2c + 37, & r = 18; & c + 52, & r = 23; \\ 2c + 8, & r = 1; & c - 6, & r = 3; & 0, & r = 13; \\ 2c + 16, & r = 4; & c + 2, & r = 20; & 12, & r = 21; \\ 2c + 33, & r = 24; & c + 12, & r = 12; & 24, & \text{otherwise,} \end{cases}$$

$$p_c = \begin{cases} 2c + 41, & r = 19; & c + 14, & r = 2, 10; \\ c + 4, & r = 21; & c + 28, & r = 22; \\ c + 11, & r = 6, 7, 8, 15, 16, 17; & c + 37, & r = 0, 1, 9, 18; \\ c + 12, & r = 13; & 25, & \text{otherwise.} \end{cases}$$

Based on these observations, we propose the following conjecture:

Conjecture 5.5. *Fix a subtraction set S. There is*

- *a positive integer q, which is a multiplier of the period of* SUB(S)*;*
- *positive integers $\alpha_r, \beta_r, \lambda_r, \mu_r$ for each $0 \le r < q$,*

such that for sufficiently large $c = tq + r$,

- *the pre-period of* SUB$(S \cup \{c\})$ *is $\ell_c = \alpha_r d + \beta_r$;*
- *the period of* SUB$(S \cup \{c\})$ *is $p_c = \lambda_r d + \mu_r$.*

Theorem 5.6. *Conjecture 5.5 holds in the following cases:*

(1) *$1 \in S$ and the element of S are all odd;*
(2) *$S = \{1, b\}$;*
(3) *$S = \{a, 2a\}$;*
(4) *$S = \{a, a + 1, \ldots, b - 1, b\}$.*

Proof. (1) The period of $\mathcal{G}_S$ is $q = 2$. If c is odd, then $\mathcal{G}_{S \cup \{c\}} = \mathcal{G}_S$. If c is even, denote by s the maximal number in S. Then

$$\mathcal{G}_{S \cup \{c\}} = (01)^{c/2} (23)^{s - 1/2} 2,$$

$\ell = 0$ and $p = d + c$.

(2) Let $S = \{1, b\}$. If b is odd, then $q = 2$,

$$\ell = 0, \quad p = \begin{cases} c + b, & r = 0; \\ 2, & r = 1 \end{cases}$$

by Examples 2.2 and 2.4. If b is even, then it follows from Examples 2.5, 2.6 and Theorem 2.7.

(3) follows from Theorem 3.1.

(4) follows from Theorem 4.1. □

6. Ultimately bipartite nim-sequences

A subtraction game (or its nim-sequence) is said to be *ultimately bipartite* if the period is 2. We know that $\mathcal{G}_S$ is ultimately bipartite with pre-period 0 if and only if $1 \in S$ and all elements in S are odd, see Example 2.2.

Example 6.1. Let $a \geq 3$ be an odd integer. If S is in one of the following cases:

- $S = \{3, 5, 9, \ldots, 2^a + 1\}$;
- $S = \{3, 5, 2^a + 1\}$;
- $S = \{a, a + 2, 2a + 3\}$;
- $S = \{a, 2a + 1, 3a\}$;

then $\textsc{sub}(S)$ is ultimately bipartite. See [CH10, Theorem 2] and [Ho15, Theorem 5].

Lemma 6.2. *If $\mathcal{G}_S$ is ultimately bipartite, then all elements in S are odd.*

Proof. As shown in [CH10, Theorem 3], there exists an integer n_0 such that for $n \geq n_0$, $\mathcal{G}(n) = 0$ if n is even; $\mathcal{G}(n) = 1$ if n is odd. Take an even number $n \geq n_0 + s_k$. Then

$$0 = \mathcal{G}(n) = \operatorname{mex}\{\mathcal{G}(n - s) : s \in S\},$$

which implies that $\mathcal{G}(n - s) = 1$ for all $s \in S$. Hence all $s \in S$ are odd. □

We have the following new ultimately bipartite subtraction sets inspired by our conjecture.

Theorem 6.3. *Let $a \geq 3$ be an odd integer and $t \geq 1$. The subtraction game $\textsc{sub}(S)$ is ultimately bipartite in the following cases:*

(1) $S = \{a, a + 2, (2a + 2)t + 1\}$;
(2) $S = \{a, 2a + 1, (3a + 1)t - 1\}$;
(3) $S = \{a, 2a - 1, (3a - 1)t + a - 2\}$.

Proof. Write $a = 2k + 1$ and $c = \max S$.

(1) When $a \geq 5, k \geq 2$, the first $(k + 1)(a + 1)(2t + 1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((a+1)(2t+1)i + j\big), 0 \leq j < (a+1)(2t+1) = c + a$
0	$0^a1, [1^{a-1}22, 0^a1]^{t-1}, 1^{a-1}22, 02^{a-3}331$
1	$030^{a-2}1, [01^{a-2}21, 020^{a-2}1]^{t-1}, 01^{a-2}21, 0202^{a-5}321$
i	$(01)^{i-1}030^{a-2i}1, [(01)^{i-1}01^{a-2i}21, (01)^{i-1}020^{a-2i}1]^{t-1}$, $(01)^{i-1}01^{a-2i}21, (01)^{i-1}0202^{a-2i-3}321$
$k-1$	$(01)^{k-2}030^31, [(01)^{k-2}01^321, (01)^{k-2}020^31]^{t-1}$, $(01)^{k-2}01^321, (01)^{k-2}020321$
k	$[(01)^{k-1}0301, (01)^{k-1}0121]^{t-1}, (01)^{k-1}03\mathbf{01}$, $(01)^{k-1}0101, (01)^{k-1}0101$

Hence the pre-period is

$$\ell = (k+1)(c+a) - 2a - 4 = (k+1)c + 2k^2 - k - 5$$

and the period is $p = 2$. The case $a = 3$ will be shown in (3).

(2) The first $(k+1)((3a+1)t + a - 1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big(((3a+1)t+a-1)i+j\big), 0 \le j < (3a+1)t+a-1 = c+a$
0	$[0^a, 1^a, 02^{a-1}, 1]^t, 3^{a-1}$
1	$[020^{a-2}, 101^{a-2}, (01)32^{a-3}, 1]^{t-1}$, $020^{a-2}, 101^{a-2}, (01)02^{a-3}, 1, (01)3^{a-3}$
i	$[(01)^{i-1}020^{a-2i}, 1(01)^{i-1}01^{a-2i}, (01)^i32^{a-2i-1}, 1]^{t-1}$, $(01)^{i-1}020^{a-2i}, 1(01)^{i-1}01^{a-2i}, (01)^i02^{a-2i-1}, 1, (01)^i3^{a-2i-1}$
$k-1$	$[(01)^{k-2}020^3, 1(01)^{k-2}01^3, (01)^{k-1}32^2, 1]^{t-1}$, $(01)^{k-2}020^3, 1(01)^{k-2}01^3, (01)^{k-1}02^2, 1, (01)^{k-1}3^2$
k	$[(01)^{k-1}02\mathbf{0}, \mathbf{1}(01)^{k-1}01, (01)^k3, 1]^{t-1}$, $(01)^{k-1}020, 1(01)^{k-1}01, (01)^k0, 1, (01)^k$

Hence the pre-period is

$$\ell = (k+1)(c+a) - 3a - 1 = (k+1)c + 2k^2 - 3k - 3$$

and the period is $p = 2$.

(3) The first $(k+1)(3a-1)(t+1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((3a-1)(t+1)i+j\big), 0 \le j < (3a-1)(t+1) = c+2a+1$
0	$[0^a1^a2^{a-1}]^t$, $0^{a-2}331^{a-3}(10)^12^{a-2}(01)^1$
1	$[0^{a-3}(01)^131^{a-3}(10)^12^{a-2}(01)^1]^t$, $0^{a-4}3(01)^131^{a-5}(10)^22^{a-4}(01)^2$
i	$[0^{a-2i-1}(01)^i31^{a-2i-1}(10)^i2^{a-2i}(01)^i]^t$, $0^{a-2i-2}3(01)^i31^{a-2i-3}(10)^{i+1}2^{a-2i-2}(01)^{i+1}$
$k-1$	$[0^2(01)^{k-1}31^2(10)^{k-1}2^3(\mathbf{01})^{k-1}]^t$, $0^13(01)^{k-1}3(10)^k2^1(01)^k$,
k	$[(01)^k3(10)^k2(01)^k]^{t-1}, (01)^{3k+1}$, $(01)^{3k+1}$

Hence the pre-period is

$$\ell = (k+1)(c+2a+1) - 2(7k+2) = (k+1)c + 4k^2 - 7k - 1$$

and the period is $p = 2$. □

Remark 6.4. One may expect that if $\textsc{sub}(a, b, c)$ is ultimately bipartite, then so is $\textsc{sub}(a, b, d)$ for sufficient large d with $d \equiv c \bmod (a + b)$. This is not true in general. For example, $\textsc{sub}(3, 11, 13)$ is ultimately bipartite but $\textsc{sub}(3, 11, 14t + 13)$ has period $14t + 16$, $t \ge 1$.

Remark 6.5. Write $a = 2k + 1$. Consider the four-elements subtraction set $S = \{a, 2a + 1, 3a, c\}$, $c > 3a$ is odd. For $3 \le a \le 25, c < 500$, we find the following phenomenon.

- If $c = 4a + 1$, then $\ell = 0$ and $p = 5a + 1$.
- If $c = (4i + 2)a - 1$ with $1 \le i < k$, then $\ell = (8i - 1)a + 2i - 1$ and $p = 4a$.
- Otherwise, $\textsc{sub}(S)$ is ultimately bipartite.

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