LINEAR ALGEBRA HOMEWORK 1

LECTURER: BONG H. LIAN

This homework offers warm-up exercises on sets: you will work with various set notions like memberships, subsets, intersections, unions, disjointness, and partitions. You will also work with maps of various kinds: injective, surjective, and bijective.

Exercise 0.1. *WRITE UP This exercise will show that 'fractions' is a way to break up a particular set $\mathcal{P}$ into smaller sets – namely fractions. Two fractions will either be disjoint or the same. For this reason, we say that the fractions form a* **partition** *of $\mathcal{P}$.*
Recall that for $a, b \in \mathbb{Z}$, $a \neq 0$, the fraction b/a is defined to be the set $\mathcal{P}$ of all pairs

$$(n, m), \quad n, m \in \mathbb{Z}, \quad n \neq 0, \text{ such that } am = bn.$$

(Recall that you can also think of this set as the set of all equations $m = nx$, $\ n, m \in \mathbb{Z}$, $\ n \neq 0$, s.t. $am = bn$, which obviously contains the same information as the pairs above.)

(1) *Prove that two such sets $b/a = b'/a'$ iff $ab' = ba'$.*
(2) *Prove that $b/a \cap b'/a' = \emptyset$ (i.e. the two sets have no members in common) iff $ab' \neq ba'$.*

(3) *Prove that the map $\iota : \mathbb{Z} \to \mathbb{Q}$, $n \mapsto n/1$ is injective. Therefore, we can treat $\mathbb{Z}$ as a subset of $\mathbb{Q}$ by treating set $n/1$ as the integer n.*
(4) *Prove that this map is not surjective, i.e. there is a fraction $b/a \in \mathbb{Q}$ which is not $\iota(c)$ for any $c \in \mathbb{Z}$.*

Exercise 0.2. *Verify that the multiplication rule given by*

$$\times : \mathbb{Q}^2 \to \mathbb{Q}, \ (b/a, b'/a') \mapsto b/a \times b'/a' := (bb')/(aa')$$

is well-defined. Note that this rule generalizes the usual rule for multiplying integers, i.e. grouping apples. Therefore, the multiplication rule for $\mathbb{Z}$ remains the same after we treat it as a subset of $\mathbb{Q}$. (See next exercise.)

(1) *Prove using this multiplication rule, that b/a is a solution to the equation $ax = b$. That is, $a \times b/a = b$.)*
(2) *Prove that this is the only solution. That is, if b'/a' is another solution, then $b'/a' = b/a$.*

Exercise 0.3. *Write For $n \in \mathbb{Z}$, put $\iota(n) = n/1$. Verify the identities*

$$\iota(1) = 1/1, \quad \iota(nn') = \iota(n)\iota(n'), \quad n, n' \in \mathbb{Z}.$$

Also express $\iota(n + n')$ in terms of $\iota(n)$, $\iota(n')$.

Exercise 0.4. *Let F be a field. For $X, Y, Z \in F^2$ and $\lambda, \mu \in F$, verify that*
V1. $(X + Y) + Z = X + (Y + Z)$
V2. $X + Y = Y + X$
V3. $X + 0 = X$
V4. $X + (-X) = 0$
V5. $\lambda(X + Y) = \lambda X + \lambda Y$
V6. $(\lambda + \mu)X = \lambda X + \mu X$
V7. $(\lambda\mu)X = \lambda(\mu X)$
V8. $1X = X$.
The same holds true for F^n.

Suggestion: *Think about exactly what facts about F you need to use to prove each of these statements.*

Exercise 0.5. *WRITE UP Recall that for a given field F, we have a* **characteristic map** *defined by*

$$\iota_F : \mathbb{Z} \to F, \quad n \mapsto n \cdot 1_F.$$

We say that F has characteristics p if there exists a smallest positive integer p such that $\iota_F(p) = 0_F$. If such a p does not exists, we say that F has characteristics 0.

- *(1) What is the characteristics of the field $\mathbb{Q}$? Prove your answer.*
- *(2) Prove that if F is finite then p exists and it must be a prime number.*
- *(3) Fix a prime number p. For each integer, put*

  $$\bar{a} = a + p\mathbb{Z} := \{a + pn | n \in \mathbb{Z}\} = \{..., a - p, a, a + p, a + 2p, ...\}.$$

  Define the set

  $$\mathbb{Z}/p := \{a + p\mathbb{Z} | a \in \mathbb{Z}\}.$$

  Consider the map

  $$f_p : [p] := \{0, 1, ..., p - 1\} \to \mathbb{Z}/p, \quad a \mapsto \bar{a} := a + p\mathbb{Z}.$$

  Show that f_p is a bijection, hence $\#\mathbb{Z}/p = p$.
- *(4) Show that the set $\mathbb{Z}/p$ can be made a field with distinguished members $\bar{0}, \bar{1}$, by giving it 4 operations $+, \times, -, 1/\cdot$. Therefore, for every prime number p, you have constructed a* **finite field** *$\mathbb{F}_p$ with p members.*

LINEAR ALGEBRA HOMEWORK 2

LECTURER: BONG H. LIAN

While you should always try all assigned problems, you should write up only the ones you are asked to write up.

Assume U, V, W are F-vector spaces.

Exercise 0.1. *If $f : U \to V$ and $g : V \to W$ are linear maps, verify that their composition $gf \equiv g \circ f : U \to W$ is also linear.*

Exercise 0.2. *Let $\mathrm{Iso}(V, V)$ be the set of isomorphisms (i.e. linear bijections) $\phi : V \to V$, and $\mathrm{Iso}(V, W)$ the set of isomorphisms $f : V \to W$. Suppose $f_0 \in \mathrm{Iso}(V, W)$. Show that the map*

$$T : \mathrm{Iso}(V, V) \to \mathrm{Iso}(V, W), \ \ \phi \mapsto f_0 \circ \phi$$

bijects, by writing the inverse map.

Exercise 0.3. *Describe $sol(E)$ to the following system E in $\mathbb{R}^4$, using row reduction and then giving an isomorphism $f : \mathbb{R}^\ell \to \ker L_A$ (including specifying the appropriate ℓ), where A is the coefficient matrix of the system:*

$$E_0 : \qquad \begin{aligned} &x + y + z + t = 0 \\ &x + y + 2z + 2t = 0 \\ &x + y + 2z - t = 0. \end{aligned}$$

Replace the 0 on the right side of first equation by 1 to get a new inhomogeneous system E_1, and then describe its solution set $sol(E_1)$ by writing down an explicit translation map.

Exercise 0.4. *WRITE UP Let E be an n-variable F-linear system. Prove that*

$$\begin{aligned} & E \textit{ is homogenous} \\ \Leftrightarrow \quad & 0 \in sol(E) \\ \Leftrightarrow \quad & sol(E) \textit{ is } F\textit{-subspace of } F^n. \end{aligned}$$

Try to make your proof as simple as possible, say less than half a page.

Exercise 0.5. *WRITE UP Let $F[x]_d$ be the F-subspace of $F[x]$ consisting of all polynomials $p(x)$ of degree at most d, i.e. the highest power*

x^n appearing in $p(x)$ is at most x^d. Consider the map

$$L_{n,d} := (1-x^2)(\frac{d}{dx})^2 - 2x\frac{d}{dx} + n(n+1)id : F[x]_d \to F[x]_d$$

for integer $n \geq 0$. Verify that $L_{n,d}$ is F-linear. Describe $\ker L_{n,d}$ *by solving the linear equation*

$$L_{n,d}(f) = 0$$

for $n = 0, 1, 2$, say by giving a basis of $\ker L_{n,d}$*. Equivalently, find $k \in \mathbb{Z}_{\geq 0}$ (which can depend on n, d) such that you can construct an F-isomorphism*

$$f : F^k \to \ker L_{n,d}.$$

LINEAR ALGEBRA HOMEWORK 3

LECTURER: BONG H. LIAN

F denotes a field. Assume U, V, W are F-vector spaces, and all dimensions are F-dimensions.

Exercise 0.1. *Let V be a F-subspace of F^n. Decide whether each of the following is TRUE of FALSE. Justify your answer. For (a)-(e), assume that* $\dim V = 3$.
(a) Any 4-tuple of V is linearly dependent.
(b) Any 2-tuple of V is linearly independent.
(c) Any 3-tuple of V is a basis.
(d) Some 3-tuple of V is a basis.
(e) V contains a linear subspace W with $\dim\ W = 2$.
(f) $(1, \pi)$, $(\pi, 1)$ *form a basis of* $\mathbb{R}^2$. *You can assume that* $|\pi - 3.14| < 0.01$.
(g) $(1, 0, 0)$, $(0, 1, 0)$ *do not form a basis of the plane* $x - y - z = 0$.
(h) $(1, 1, 0)$, $(1, 0, 1)$ *form a basis of the plane* $x - y - z = 0$.
(i) If A is a 3×4 matrix, then the subspace V of F^4 generated by the rows of A is at most 3 dimensional.
(j) If A is a 4×3 matrix, then the subspace V of F^3 generated by the rows of A is at most 3 dimensional.

Exercise 0.2. *WRITE UP Let*

$$V = \{(a + b, a, c, b + c) | a, b, c \in F\} \subset F^4.$$

Verify that V is an F-subspace of F^4. Find a basis of V.

Exercise 0.3. *Fix $0 < k < n$ and consider the decomposition*

$$F^n \equiv F^k \oplus F^{n-k}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \equiv \begin{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \\ \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} u_k \\ \ell_{n-k} \end{bmatrix}.$$

Show if $A \in M_{n,n}$, then A also 'decomposes' into a corresponding block form

$$A \equiv \begin{bmatrix} P_{k,k} & Q_{k,n-k} \\ R_{n-k,k} & S_{n-k,n-k} \end{bmatrix}$$

so that the column vector Ax can be expressed in terms of the column vectors $Pu, Q\ell, Ru, S\ell$. If you are confused, do the special case $n = 3, k = 1$ first.

Exercise 0.4. *WRITE UP In 1 line, prove that every matrix $A \in M_{n,n}$ satisfies a nontrivial polynomial equation of the form*

$$a_0 I_n + a_1 A + \cdots + a_k A^k = 0.$$

Exercise 0.5. *Find a basis of* $sol(E)$ *in* F^4 *for*

$$E: \quad x - y + 2z + t = 0.$$

Exercise 0.6. *Find a basis for each of the subspaces* $\ker L_A$ *and* $\operatorname{im} L_A$ *of* F^4*, where* A *is the matrix*

$$\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & -2 & 4 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 4 & -5 & -1 \end{bmatrix}.$$

Exercise 0.7. *We know that* $V^2 = V \times V$ *form a vector space. Define an* F*-vector space structure on* $U \times V$ *a vector space. Let's call it the* **direct sum** $U \oplus V$ *of* U, V*. If* $\dim U = k$ *and* $\dim V = n$*, what is* $\dim(U \oplus V)$*? Prove your assertion in 5 lines.*

Exercise 0.8. *(Revisit MMC) We specialize to the case* $V = F^2$*. Let* $(v_1, v_2) \in V^2$*, put* $A = [v_1, v_2] \in M_{2,2}$*, and write* $v_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$*,* $v_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$*.*
(a) (A numerical test for isomorphism) Show that v_1, v_2 *are 'parallel', i.e. one a scalar multiple of the other, iff they are dependent, iff*

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

iff L_A *is not an isomorphism, iff* (v_1, v_2) *is not a basis of* V*.*
(b) Now suppose L_A *is an isomorphism. Can you find the matrix* B *corresponding to (under MMC) to the inverse isomorphism* $L_A^{-1} : F^2 \to F^2$*?*

LINEAR ALGEBRA HOMEWORK 4

LECTURER: BONG H. LIAN

Assume U, V, W are F-vector spaces. Put $\operatorname{End} V = \operatorname{Hom}(V, V)$.

Exercise 0.1. *Find the dimension of $M_{2,2}$ by giving a basis of this vector space. Generalize your result to $M_{k,l}$.*

Exercise 0.2. *Let $f, g : V \to V$ be two given maps such that $f \circ g = id_V$.*
(a) Show that g is injective and f is surjective.
(b) Assume in addition that $\dim V < +\infty$ and f is linear. Show that f is injective, hence g is surjective. (Hint: Use COD.)
(c) Conclude that g is bijective, and that $f = g^{-1}$ and $g \circ f = id_V$.
(d) Let $A, B \in M_{n,n}$. Show that if $AB = I$, then $BA = I$.

Exercise 0.3. *Another proof. Show that if $\ker(BA) = (0)$ then $\ker A = (0)$, hence A is an isomorphism. Conclude that $B = A^{-1}$. (Hint: COD.)*

Exercise 0.4. WRITE UP *Prove that for $A \in M_{n,n}$, $\det A^t = \det A$. You will need the fact that $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$ for any bijection of $\{1, 2, .., n\}$.*

Exercise 0.5. *Decide if $A = [e_3, e_1+e_2, e_2] \in M_{3,3}$ is invertible. If so, compute A^{-1}. Here e_i are the standard unit vectors if F^3.*

Exercise 0.6. WRITE UP *Let $U \subset V$ be a subspace and $x \in \operatorname{End} V$ such that $xU \subset U$. In 5 lines, prove that there is a canonical map*

$$\bar{x} : V/U \to V/U, \quad v + U \mapsto xv + U.$$

That is check that this is well-defined. Show it satisfies the following: if $p(t) \in F[t]$, and $p(x) = 0$ in $\operatorname{End} V$ then $p(\bar{x}) = 0$ in $\operatorname{End} V/U$.

Exercise 0.7. *By row reduction, compute*

$$\det[e_3 + e_2 + e_1, e_1 + e_2, e_2].$$

Redo this it by using linearity of $\det$ in each column.

Exercise 0.8. *Assume that $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}$ is invertible. Find a formula for A^{-1}. That is to say, find each entry of A^{-1} in terms of the 4 entries a_{ij} of A. Be sure to check that you do get $AA^{-1} = A^{-1}A = I$. From this, can you guess the answer for 3×3 matrices.*

Exercise 0.9. WRITE UP *Prove that the minimal polynomial of a matrix $A \in M_{n,n}$ is conjugation invariant, i.e. $\mu_{g^{-1}Ag}(x) = \mu_A(x)$ for all $g \in \operatorname{Aut}_n$. Conclude that the algebra $F[x]/\mu_A(x)F[x]$ does not change under conjugations of A.*

Exercise 0.10. *Compute the* $\mu_A(x)$ *for* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

LINEAR ALGEBRA HOMEWORK 5

LECTURER: BONG H. LIAN

Exercise 0.1. *WRITE UP Find all conjugacy classes of solutions to the matrix equation*

$$X^3 = 0$$

in $M_3(\mathbb{C})$.

Exercise 0.2. *Work out the structure of the* $\mathbb{Z}$*-module* $M = \mathbb{Z}^3/A\mathbb{Z}^2$ *where*

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix}.$$

Namely determine its free part and torsion part of M.

Exercise 0.3. *WRITE UP Take* $R = \mathbb{C}[t]$, *the polynomial algebra with complex coefficientgs. Work out the structure of the* R*-module* $M = R^3/AR^2$ *where*

$$A = \begin{bmatrix} t^2 & 2t \\ 0 & t \\ t^3 - 1 & t - 2 \end{bmatrix}.$$

Namely determine its free part and torsion part of M.

Exercise 0.4. *WRITE UP Show that there is a one to one correspondence between the isomorphism classes of* n*-dimensional* $F[t]$*-modules and the conjugacy classes of* $n \times n$ *matrices.*

LINEAR ALGEBRA HOMEWORK 5

LECTURER: BONG H. LIAN

Exercise 0.1. *Let $x, y \in M_{n,n}$. Recall that x, y are conjugates of each other iff there exists an invertible matrix g such that $y = g^{-1}xg$. Prove your assertions.*
(a) Suppose $\det x \neq \det y$. Can x, y be translates of each other, i.e. can $[x] = [y]$?
(b) Suppose $\det x = \det y$. Does this imply that $[x] = [y]$?

Exercise 0.2. *Prove that if $x \in M_{n,n}$ has characteristic polynomial $p_x(t)$ which has n distinct roots, then x is diagonalizable.*

Exercise 0.3. *WRITE UP Let $x_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and let $[x_0]$ be its conjugation class in $M_{2,2}$. Show that there is a surjective map*

$$\pi : [x_0] \to \mathbb{P}^1 := \text{the set of all lines in } F^2$$

given by $x \mapsto \ker x$. Here, a line in F^2 is a one dimensional subspace of F^2. Can you describe the subset

$$\pi^{-1}(\ker x) = \{y \in [x_0] | \ker y = \ker x\}$$

for each x? Prove your assertions.

Exercise 0.4. *WRITE UP Let A be an F-algebra and V be a finite dimensional A-space. Show that V is a quotient A-space of a direct sum $A^{\oplus k}$ of k copies of A, regarded as an A-space. In other words, there exists a surjective A-space homomorphism*

$$\pi : A^{\oplus k} \twoheadrightarrow V.$$

We say that an A-space M is semi-minimal if it decomposes into a independent sum of A-subspaces which are minimal. Show that if A is semi-minimal as an A-space, then any A-space V is semi-minimal.

2021 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

LECTURER: BONG H. LIAN

This is a preliminary version of the research projects, subject to updates later.

0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). For simplicity, you can think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, \ldots$. Likewise a linear map means an F-linear, and an F-matrix means a matrix with entries in F. Put

$$\begin{aligned}
\mathrm{Hom}(U,V) &:= \text{the set of all linear maps } U \to V \\
\mathrm{End}\,V &:= \mathrm{Hom}(V,V), \quad \text{the algebra of linear maps } V \to V \\
\mathrm{Aut}\,V &:= \{f \in \mathrm{End}\,V | f \text{ is bijective}\} \\
\mathrm{Aut}\,_nF &:= \mathrm{Aut}\,F^n \quad \text{the set of all isomorphisms } F^n \to F^n. \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n \ F\text{-matrices} \\
& \text{with the usual matrix product} \\
I &\equiv I_n := [e_1, .., e_n], \quad \text{the identity matrix in } M_n
\end{aligned}$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project 1

We say that two matrices $x, x' \in M_n(F)$ are **conjugate** if

$$\boxed{x' = gxg^{-1}}$$

for some $g \in \mathrm{Aut}_n F$.

Problem 1.1.

Solve the matrix equation

$$\boxed{x^2 = I}$$

in the 2×2 matrix algebra $M_2(F)$ up to conjugation. In other words, classify solutions to the equations up to conjugation by $\mathrm{Aut}_2 F$. Thus two solutions are considered equivalent if they are conjugate of each other. How would you describe a 'nice' matrix x that represent each equivalence class of solutions to the equation? This same notion of solving a matrix equation shall apply to the next two problems as well.

Do the same for the matrix equation

$$\boxed{x^2 = 0}$$

Problem 1.2.

Generalize Problem 1.1 by solving each of the equations

$$\boxed{x^k = I, \quad k = 2, 3, ...}$$

in the matrix algebra $M_n(F)$. Do the same for

$$\boxed{x^k = 0, \quad k = 2, 3, ...}$$

Can you say any thing more in these problems when F is assumed to be a finite field of prime characteristic p?

Problem 1.3.

For $F = \mathbb{C}$, solve the matrix equation

$$\boxed{\exp(x) = I}$$

in the 2×2 matrix algebra $M_2(F)$ up to conjugation. You may assume that $exp(x)$ is the limit (in the sense of calculus) of the sequences of matrices:

$$I, \quad I + x, \quad I + x + \frac{x^2}{2!}, \quad \cdots$$

with respective to the length function $||A|| = \max_{ij} |a_{ij}|$ on matrices.

Problem 1.4.

Generalize Problem 1.3 to the case of matrix algebra $M_n(F)$ for $F = \mathbb{C}$.

2021 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

LECTURER: BONG H. LIAN

This is a *preliminary* version of the research projects, subject to updates later.

0. Basic Assumptions and Notation

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$$
\begin{aligned}
\mathrm{Hom}(U,V) &:= \text{the set of all linear maps } U \to V \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \text{the algebra of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V \mid f \text{ is bijective}\} \\
\mathrm{Aut}_n F &:= \mathrm{Aut}\, F^n \ \text{ the set of all isomorphisms } F^n \to F^n. \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n \ F\text{-matrices} \\
&\ \text{with the usual matrix product} \\
U_n &:= \{A = (a_{ij}) \in M_n \mid A \text{ is upper triangular, i.e., } a_{ij} = 0 \text{ for } i > j\} \\
I \equiv I_n &:= [e_1, \cdots, e_n] \text{ the identity matrix in } M_n
\end{aligned}
$$

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project II

Problem 1.1. *Classify all F-algebra homomorphisms $M_n \to F[x]$.*

Problem 1.2. *Classify all F-algebra homomorphisms $U_n \to F[x]$.*

Recall the definition of rings and ring homomorphisms. Do the following problems.

Problem 1.3. *Classify all ring homomorphisms $M_n \to F[x]$.*

Problem 1.4. *Classify all ring homomorphisms $U_n \to F[x]$.*

Problem 1.5. *What can you say about these problems when F is a finite field of prime characteristic p?*

2021 MATHCAMP LINEAR ALGEBRA FINAL EXAM

14:00 – 15:45, JULY 31^{ST}, 2021

Note. Do all the three problems. Please write down all the details about your arguments and computations as clear as possible; no partial credit will be given. In what follows, unless otherwise stated, F will be a field.

Problem 1. *Answer the following questions.*

(1) Let

$$f(x,y,z,w) = \begin{vmatrix} x & y & z & w \\ w & x & y & z \\ z & w & x & y \\ y & z & w & x \end{vmatrix}.$$

Regarding $f(x,y,z,w)$ as a polynomial over $\mathbb{C}$, compute the coefficient of the following monomials in $f(x,y,z,w)$:

(a) x^4.

(b) x^2yz.

(2) Let $X = (x_1, \ldots, x_n)$ be a vector in F^n with $\|X\|^2 := \sum_{i=1}^n x_i^2 \neq 0$. Put

$$A = (a_{ij}) := \left(\delta_{ij} - \frac{2x_ix_j}{\|X\|^2}\right) \in M_n(F).$$

Show that $\det(A) = -1$. Here δ_{ij} denotes the Kronecker delta, i.e., $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

(3) Let $A \in M_n(F)$ such that $A^k = 0$ for some $k > 0$. Show that any element in $F[A]$ of the form

$$a_0I_n + a_1A + \cdots + a_mA^m, \ a_j \in F$$

with $a_0 \neq 0$ is invertible in $M_n(F)$.

Problem 2. *Let V be an n-dimensional vector space over F. The set of all F-linear maps from V to itself is denoted by $\mathrm{End}_F(V)$. Let $f \in \mathrm{End}_F(V)$. The expansion*

$$\det(t \cdot I_n - f) = \sum_{k=0}^{n} a_k(f)t^k.$$

defines maps $a_k \colon \mathrm{End}_F(V) \to F$. For f, $g \in \mathrm{End}_F(V)$, write $fg \equiv f \circ g$.

(1) Show that $a_k(f)$ is invariant under conjugation, i.e., for any $g \in \mathrm{Aut}_F(V)$, we have $a_k(g^{-1}fg) = a_k(f)$.

(2) Show that $a_{n-1}(fg) = a_{n-1}(gf)$ for any $f, g \in \mathrm{End}_F(V)$.

(3) Assume that $\mathrm{char}(F) = 0$. For $f \in \mathrm{End}_F(V)$, can we find an element $g \in \mathrm{End}_F(V)$ such that $fg - gf = \mathrm{id}_V$? Justify your answer.

Problem 3. *Recall that for F-vector spaces U and W, $\mathrm{Hom}_F(U, W)$ is the set of all F-linear maps from U to W. Let V be a finite-dimensional vector space over F. For any element $f \in \mathrm{Hom}_F(U, W)$, we define a map $\Phi(f) \colon \mathrm{Hom}_F(V, U) \to \mathrm{Hom}_F(V, W)$ via*

$$\Phi(f)(x) = f \circ x, \text{ for } x \in \mathrm{Hom}_F(V, U).$$

(1) Show that $\Phi(f)$ is well-defined, i.e., $\Phi(f)(x)$ is an F-linear map from V to W for any $f \in \mathrm{Hom}_F(U, W)$ and any $x \in \mathrm{Hom}_F(V, U)$.

(2) Show that the map

$$\Phi \colon \mathrm{Hom}_F(U, W) \to \mathrm{Hom}_F(\mathrm{Hom}_F(V, U), \mathrm{Hom}_F(V, W))$$

is F-linear.

(3) Given a sequence of vector spaces V_1, V_2, V_3 and linear maps f and g

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$$

such that f is injective, g is surjective, and $\ker(g) = \mathrm{im}(f)$, show that for any vector space V, the induced sequence

$$\mathrm{Hom}_F(V, V_1) \xrightarrow{\Phi(f)} \mathrm{Hom}_F(V, V_2) \xrightarrow{\Phi(g)} \mathrm{Hom}_F(V, V_3)$$

has the following properties:

(a) $\Phi(f)$ is injective,

(b) $\Phi(g)$ is surjective,

(c) $\ker(\Phi(g)) = \mathrm{im}(\Phi(f))$.

LINEAR ALGEBRA HOMEWORK 1

LECTURER: BONG H. LIAN

This homework offers warm-up exercises on sets: you will work with various set notions like memberships, subsets, intersections, unions, disjointness, and partitions. You will also work with maps of various kinds: injective, surjective, and bijective.

Exercise 0.1. *WRITE UP This exercise will show that 'fractions' is a way to break up a particular set*

$$\mathcal{P} \equiv \widetilde{\mathbb{Z}^2} := \{(a,b) \in \mathbb{Z}^2 | b \neq 0\}$$

into smaller subsets – namely fractions. A fraction is of the form

$$a/b := \{(m,n) \in \mathcal{P} | an = bm\}.$$

Prove that two fractions are either equal or disjoint. For this reason, we say that the fractions form a **partition** *of* $\mathcal{P}$.

(1) *Prove that* $a/b = a'/b'$ *iff* $ab' = ba'$.
(2) *Prove that* $a/b \cap a'/b' = \emptyset$ *(i.e. the two sets have no members in common) iff* $ab' \neq ba'$.

(3) *Prove that the map* $\iota : \mathbb{Z} \to \mathbb{Q}$, $n \mapsto n/1$ *is injective. Therefore, we can treat* $\mathbb{Z}$ *as a subset of* $\mathbb{Q}$ *by treating set* $n/1$ *as the integer* n.
(4) *Prove that this map is not surjective, i.e. there is a fraction* $b/a \in \mathbb{Q}$ *which is not* $\iota(c)$ *for any* $c \in \mathbb{Z}$.

Exercise 0.2. *Verify that the multiplication rule given by*

$$\times : \mathbb{Q}^2 \to \mathbb{Q}, \ (a/b, a'/b') \mapsto a/b \times a'/b' := (aa')/(bb')$$

is well-defined. Note that this rule generalizes the usual rule for multiplying integers, i.e. grouping apples. Therefore, the multiplication rule for $\mathbb{Z}$ *remains the same after we treat it as a subset of* $\mathbb{Q}$. *(See next exercise.)*

(1) *Prove using this multiplication rule, that* a/b *is a solution to the equation* $bx = a$. *That is,* $b \times a/b = a$.*)*
(2) *Prove that this is the only solution. That is, if* a'/b' *is another solution, then* $a'/b' = a/b$.

Exercise 0.3. *Write For* $n \in \mathbb{Z}$, *put* $\iota(n) = n/1$. *Verify the identities*

$$\iota(1) = 1/1, \quad \iota(nn') = \iota(n)\iota(n'), \quad n, n' \in \mathbb{Z}.$$

Also express $\iota(n + n')$ *in terms of* $\iota(n)$, $\iota(n')$.

Exercise 0.4. *Let F be a field. For $X, Y, Z \in F^2$ and $\lambda, \mu \in F$, verify that*
V1. $(X+Y)+Z = X+(Y+Z)$
V2. $X+Y = Y+X$
V3. $X+0 = X$
V4. $X+(-X) = 0$
V5. $\lambda(X+Y) = \lambda X + \lambda Y$
V6. $(\lambda+\mu)X = \lambda X + \mu X$
V7. $(\lambda\mu)X = \lambda(\mu X)$
V8. $1X = X$.
Here $\lambda(x_1, x_2) := (\lambda x_1, \lambda x_2)$. The same holds true for F^n, the n-times Cartesian product $F^n = F \times \cdots \times F$.
Suggestion: *Think about exactly what facts about F you need to use to prove each of these statements.*

Exercise 0.5. *WRITE UP Recall that for a given field F, we have a* **characteristic map** *defined by*

$$\iota_F : \mathbb{Z} \to F, \quad n \mapsto n \cdot 1_F.$$

We say that F has characteristics p if there exists a smallest positive integer p such that $\iota_F(p) = 0_F$. If such a p does not exists, we say that F has characteristics 0.

- *(1) What is the characteristics of the field $\mathbb{Q}$? Prove your answer.*
- *(2) Prove that if F is finite then p exists and it must be a prime number.*
- *(3) Fix a prime number p. For each integer, put*

  $$\bar{a} = a + p\mathbb{Z} := \{a + pn | n \in \mathbb{Z}\} = \{..., a-p, a, a+p, a+2p, ...\}.$$

  Define the set

  $$\mathbb{Z}/p := \{a + p\mathbb{Z} | a \in \mathbb{Z}\}.$$

  Consider the map

  $$f_p : [p] := \{0, 1, ..., p-1\} \to \mathbb{Z}/p, \quad a \mapsto \bar{a} := a + p\mathbb{Z}.$$

  Show that f_p is a bijection, hence $\#\mathbb{Z}/p = p$.
- *(4) Show that the set $\mathbb{Z}/p$ can be made a field with distinguished members $\bar{0}, \bar{1}$, by giving it 4 operations $+, \times, -, 1/\cdot$. Therefore, for every prime number p, you have constructed a* **finite field** *$\mathbb{F}_p$ with p members.*

LINEAR ALGEBRA HOMEWORK 2

LECTURER: BONG H. LIAN

While you should always try all assigned problems, you should write up only the ones you are asked to write up.

Exercise 0.1. WRITE UP *Let F be a field. Recall that E a subfield of F if E is subset of F that is closed under the 4 algebraic operations of F $(+, -, \times, 1/)$ and contains $0, 1$ of F. Now suppose E is a subfield of F, and S is a subset of F. In 10 lines, prove that there exists a unique subfield $E(S)$ with the following properties: $E(S)$ contains S; if K is any subfield of F containing E and S, then K contains $E(S)$. In other words $E(S)$ is the 'smallest' subfield F containing E and S.*

Exercise 0.2. *Show that if F is a field of characteristic 0, then F contains the field $\mathbb{Q}$ as a subfield in some sense. Make precise in what sense this is true. Similarly, if F is a field of characteristic prime p, then F contains the field $\mathbb{F}_p$ of p elements as a subfield.*

Exercise 0.3. WRITE UP *Consider the field $F = \mathbb{R}$, and $\alpha \in \mathbb{R}$ a root of a polynomial with coefficients in $\mathbb{Q}$, say*

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d$$

where $a_0, .., a_d \in \mathbb{Q}$ and $a_d \neq 0$. Let $S = \{\alpha\}$. Can you describe the field $\mathbb{Q}(S)$ in terms of α?
Idea: Does $\mathbb{Q}(S)$ contain $1, \alpha, \alpha^2, \ldots$?

Assume U, V, W are F-vector spaces.

Exercise 0.4. *If $f : U \to V$ and $g : V \to W$ are linear maps, verify that their composition $gf \equiv g \circ f : U \to W$ is also linear.*

Exercise 0.5. *Let $\mathrm{Iso}(V, V)$ be the set of isomorphisms (i.e. linear bijections) $\phi : V \to V$, and $\mathrm{Iso}(V, W)$ the set of isomorphisms $f : V \to W$. Suppose $f_0 \in \mathrm{Iso}(V, W)$. Show that the map*

$$T : \mathrm{Iso}(V, V) \to \mathrm{Iso}(V, W), \ \phi \mapsto f_0 \circ \phi$$

bijects, by writing the inverse map.

Exercise 0.6. *Describe $sol(E)$ to the following system E in $\mathbb{R}^4$, using row reduction and then giving an isomorphism $f : \mathbb{R}^\ell \to \ker L_A$ (including specifying the appropriate ℓ), where A is the coefficient matrix of the system:*

$$E_0 : \quad \begin{aligned} x+y+z+t&=0\\ x+y+2z+2t&=0\\ x+y+2z-t&=0. \end{aligned}$$

Replace the 0 on the right side of first equation by 1 to get a new inhomogeneous system E_1, and then describe its solution set $sol(E_1)$ by writing down an explicit translation map.

Exercise 0.7. *Let E be an n-variable F-linear system. Prove that*

$$\begin{aligned} &E \text{ is homogenous}\\ \Leftrightarrow\ & 0 \in sol(E)\\ \Leftrightarrow\ & sol(E) \text{ is } F\text{-subspace of } F^n. \end{aligned}$$

Try to make your proof as simple as possible, say less than half a page.

Exercise 0.8. *Let $F[x]_d$ be the F-subspace of $F[x]$ consisting of all polynomials $p(x)$ of degree at most d, i.e. the highest power x^n appearing in $p(x)$ is at most x^d. Consider the map*

$$L_{n,d} := (1-x^2)(\frac{d}{dx})^2 - 2x\frac{d}{dx} + n(n+1)id : F[x]_d \to F[x]_d$$

for integer $n \geq 0$. Verify that $L_{n,d}$ is F-linear. Describe $\ker L_{n,d}$ *by solving the linear equation*

$$L_{n,d}(f) = 0$$

for $n = 0, 1, 2$, say by giving a basis of $\ker L_{n,d}$. *Equivalently, find $k \in \mathbb{Z}_{\geq 0}$ (which can depend on n, d) such that you can construct an F-isomorphism*

$$f : F^k \to \ker L_{n,d}.$$

LINEAR ALGEBRA HOMEWORK 3

LECTURER: BONG H. LIAN

F denotes a field. Assume U, V, W are F-vector spaces, and all dimensions are F-dimensions.

Exercise 0.1. *Let V be a F-subspace of F^n. Decide whether each of the following is TRUE of FALSE. Justify your answer. For (a)-(e), assume that* $\dim V = 3$.
(a) Any 4-tuple of V is linearly dependent.
(b) Any 2-tuple of V is linearly independent.
(c) Any 3-tuple of V is a basis.
(d) Some 3-tuple of V is a basis.
(e) V contains a linear subspace W with $\dim\ W = 2$.
(f) $(1, \pi)$, $(\pi, 1)$ form a basis of $\mathbb{R}^2$. You can assume that $|\pi - 3.14| < 0.01$.
(g) $(1, 0, 0)$, $(0, 1, 0)$ do not form a basis of the plane $x - y - z = 0$.
(h) $(1, 1, 0)$, $(1, 0, 1)$ form a basis of the plane $x - y - z = 0$.
(i) If A is a 3×4 matrix, then the subspace V of F^4 generated by the rows of A is at most 3 dimensional.
(j) If A is a 4×3 matrix, then the subspace V of F^3 generated by the rows of A is at most 3 dimensional.

Exercise 0.2. WRITE UP *Let*

$$V = \{(a + b, a, c, b + c) | a, b, c \in F\} \subset F^4.$$

Verify that V is an F-subspace of F^4. Find a basis of V.

Exercise 0.3. *Fix $0 < k < n$ and consider the decomposition*

$$F^n \equiv F^k \oplus F^{n-k}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \equiv \begin{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \\ \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} u_k \\ \ell_{n-k} \end{bmatrix}.$$

Show if $A \in M_{n,n}$, then A also 'decomposes' into a corresponding block form

$$A \equiv \begin{bmatrix} P_{k,k} & Q_{k,n-k} \\ R_{n-k,k} & S_{n-k,n-k} \end{bmatrix}$$

so that the column vector Ax can be expressed in terms of the column vectors $Pu, Q\ell, Ru, S\ell$. If you are confused, do the special case $n = 3, k = 1$ first.

Exercise 0.4. WRITE UP *Let $A \in M_{n,n}(F)$. In 2 lines, prove that the F-algebra $F[A]$ of polynomials of A has dimension at most n^2. Conclude that A satisfies a nontrivial polynomial equation of the form*

$$a_0 I_n + a_1 A + \cdots + a_k A^k = 0.$$

Exercise 0.5. *Find a basis of* $sol(E)$ *in* F^4 *for*

$$E: \quad x - y + 2z + t = 0.$$

Exercise 0.6. *Find a basis for each of the subspaces* $\ker L_A$ *and* $\operatorname{im} L_A$ *of* F^4, *where* A *is the matrix*

$$\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & -2 & 4 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 4 & -5 & -1 \end{bmatrix}.$$

Exercise 0.7. *We know that* $V^2 = V \times V$ *form a vector space. Define an* F*-vector space structure on* $U \times V$ *a vector space. Let's call it the* **direct sum** $U \oplus V$ *of* U, V. *If* $\dim U = k$ *and* $\dim V = n$, *what is* $\dim(U \oplus V)$*? Prove your assertion in 5 lines.*

Exercise 0.8. *(Revisit MMC) We specialize to the case* $V = F^2$. *Let* $(v_1, v_2) \in V^2$, *put* $A = [v_1, v_2] \in M_{2,2}$, *and write* $v_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$, $v_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$.
(a) (A numerical test for isomorphism) Show that v_1, v_2 *are 'parallel', i.e. one a scalar multiple of the other, iff they are dependent, iff*

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

iff L_A *is not an isomorphism, iff* (v_1, v_2) *is not a basis of* V.
(b) Now suppose L_A *is an isomorphism. Can you find the matrix* B *corresponding to (under MMC) to the inverse isomorphism* $L_A^{-1} : F^2 \to F^2$*?*

Exercise 0.9. *WRITE UP Let* F *be a field and* E *a subfield of* F.
(a) Show how you can make F *an* E*-space.*
(b) Assume that $F = E(\alpha)$ *where* $\alpha \in F$ *satisfies a polynomial equation*

$$p(\alpha) = 0$$

of degree $d \geq 0$, *and that* d *is the smallest such integer. Prove that* $\dim_E F = d$ *by giving a* E*-basis of* F.
(c) Let V *be an* F*-space. Show how you can make* V *an* E*-space.*
(d) Compute $\dim_E V$ *in terms of* $\dim_E F$ *and* $\dim_F V$ *which you can assume both finite.*

LINEAR ALGEBRA HOMEWORK 4

LECTURER: BONG H. LIAN

Assume U, V, W are F-vector spaces. Put $\operatorname{End} V = \operatorname{Hom}(V, V)$.

Exercise 0.1. *Find the dimension of $M_{2,2}$ by giving a basis of this vector space. Generalize your result to $M_{k,l}$.*

Exercise 0.2. *Let $f, g : V \to V$ be two given maps such that $f \circ g = id_V$.*
(a) Show that g is injective and f is surjective.
(b) Assume in addition that $\dim V < +\infty$ and f is linear. Show that f is injective, hence g is surjective. (Hint: Use COD.)
(c) Conclude that g is bijective, and that $f = g^{-1}$ and $g \circ f = id_V$.
(d) Let $A, B \in M_{n,n}$. Show that if $AB = I$, then $BA = I$.
(e) Second proof. Show that if $\ker(BA) = (0)$ then $\ker A = (0)$, hence A is an isomorphism. Conclude that $B = A^{-1}$. (Hint: COD.)

Exercise 0.3. *WRITE UP Prove that for $A \in M_{n,n}$, $\det A^t = \det A$. You will need the fact that $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$ for any bijection of $\{1, 2, .., n\}$.*

Exercise 0.4. *Decide if $A = [e_3, e_1+e_2, e_2] \in M_{3,3}$ is invertible. If so, compute A^{-1}. Here e_i are the standard unit vectors if F^3.*

Exercise 0.5. *Let $U \subset V$ be a subspace and $x \in \operatorname{End} V$ such that $xU \subset U$. In 5 lines, prove that there is a canonical map*

$$\bar{x} : V/U \to V/U, \quad v + U \mapsto xv + U.$$

That is check that this is well-defined. Show it satisfies the following: if $p(t) \in F[t]$, and $p(x) = 0$ in $\operatorname{End} V$ then $p(\bar{x}) = 0$ in $\operatorname{End} V/U$.

Exercise 0.6. *By row reduction, compute*

$$\det[e_3 + e_2 + e_1, e_1 + e_2, e_2].$$

Redo this it by using linearity of $\det$ in each column.

Exercise 0.7. *Assume that $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}$ is invertible. Find a formula for A^{-1}. That is to say, find each entry of A^{-1} in terms of the 4 entries a_{ij} of A. Be sure to check that you do get $AA^{-1} = A^{-1}A = I$. From this, can you guess the answer for 3×3 matrices.*

Exercise 0.8. *WRITE UP Prove that the minimal polynomial of a matrix $A \in M_{n,n}$ is conjugation invariant, i.e. $\mu_{g^{-1}Ag}(x) = \mu_A(x)$ for all $g \in \operatorname{Aut}_n$. Conclude that the algebra $F[x]/\mu_A(x)F[x]$ does not change under conjugations of A.*

Exercise 0.9. *Compute the $\mu_A(x)$ for $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.*

Exercise 0.10. *WRITE UP For a given $A \in M_{m\times n}$, propose an algorithm to compute a basis of* $\ker A$ *and* $\operatorname{im} A$ *by row operations. In 10 lines prove that your algorithm is correct.*

LINEAR ALGEBRA HOMEWORK 5

LECTURER: BONG H. LIAN

Exercise 0.1. *WRITE UP* *Let $p(t) = t^2 - 3t + 1$. Take $F = \mathbb{C}$. Classify the solutions of $p(x) = 0$ in $M_{n\times n}(\mathbb{C})$.*

Exercise 0.2. *('Splitting' a map) Given any linear map $f : V \to W$, show that there exist subspaces $V' \subset V$, $W' \subset W$ such that*

$$f : V = \ker f + V' \to \operatorname{im} f + W'$$

maps $V' \xrightarrow{\cong} \operatorname{im} f$, and that both sums are independent sums.

Exercise 0.3. *WRITE UP* *Find all conjugacy classes of solutions to the matrix equation*

$$X^3 = 0$$

in $M_3(\mathbb{C})$.

Exercise 0.4. *WRITE UP* *Let A be an F-algebra and V be a finite dimensional A-space. Show that V is a quotient A-space of a direct sum $A^{\oplus k}$ of k copies of A, regarded as an A-space. In other words, there exists a surjective A-space homomorphism*

$$\pi : A^{\oplus k} \twoheadrightarrow V.$$

We say that an A-space M is semi-minimal if it decomposes into a independent sum of A-subspaces which are minimal. Show that if A is semi-minimal as an A-space, then any A-space V is semi-minimal.

Exercise 0.5. *Let $x, y \in M_{n,n}$. Recall that x, y are conjugates of each other iff there exists an invertible matrix g such that $y = g^{-1}xg$. Prove your assertions.*
(a) Suppose $\det x \neq \det y$. Can x, y be conjugates of each other, i.e. can $[x] = [y]$?
(b) Suppose $\det x = \det y$. Does this imply that $[x] = [y]$?

LINEAR ALGEBRA HOMEWORK 6

LECTURER: BONG H. LIAN

Exercise 0.1. *Decide if* $A = [e_3, e_1 + e_2, e_2] \in M_3$ *is invertible. If so, compute* A^{-1}.

Exercise 0.2. *You will prove that there is a bijection between the set of conjugation classes of* $n \times n$ F*-matrices, and the set of isomorphism classes of* $F[t]$*-spaces* V *of* $dim_F V = n$*. To each matrix* x*, define the* $F[t]$*-space by the* F*-algebra homomorphism*

$$\varphi_x : F[t] \to \operatorname{End} F^n \equiv M_{n\times n}, \quad t \mapsto x.$$

Argue if $h \in \mathrm{Aut}_n$*, then* $\varphi_{h^{-1}xh}$ *defines an isomorphic* $F[t]$*-space. Verify that the correspondence* $[x] \mapsto [\varphi_x]$ *is a bijection from conjugation classes of matrices to isomorphism classes of* $F[t]$*-spaces.*

Exercise 0.3. *WRITE UP For* $X \in M_n$*, put* $k(X) := \dim\ker X$*. Assume* $X^2 = 0$.
(a) Show that $k(X) \geq n/2$.
In less than 1 page, show that the following:
(a) A conjugation class $[X]$ *in* $\mathrm{sol}(X^2 = 0)$ *in* M_n *is uniquely determined by* $k(X)$.
(b) Given any integer $k \geq n/2$*, there is a unique conjugation class* $[X]$ *of such solutions such that* $k(X) = k$.
After doing this right, you will be quite close to solving Project 3, Problems 1 and 2.

Exercise 0.4. *WRITE UP Let* $C_\bullet$ *be a complex of* F*-spaces with* $C_0 = F^2$*,* $C_1 = F^3$*, and* $C_j = 0$ *for all* $j \neq 0, 1$*. Decide which of the following homology of* $C_\bullet$ *is possible:*
(a) $H_0 = F$*,* $H_1 = F^2$.
(b) $H_0 = F^2$*,* $H_1 = F^3$.
(c) $H_0 = F$*,* $H_1 = F$.
(d) $H_0 = 0$*,* $H_1 = F^2$.

2022 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

LECTURER: BONG H. LIAN

This is a preliminary version of the research projects, subject to change later.

0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). For simplicity, you can think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, \ldots$. Likewise a linear map means an F-linear, and an F-matrix means a matrix with entries in F. Put

$$\begin{aligned}
\mathrm{Hom}(U,V) &:= \{\text{linear maps } U \to V\} \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \ \ \text{the algebra of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V | f \text{ is bijective}\} \\
\mathrm{Aut}\,_n F &:= \mathrm{Aut}\, F^n \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n \ F\text{-matrices} \\
& \text{with the usual matrix product} \\
I &\equiv I_n := [e_1, .., e_n], \ \ \text{the identity matrix in } M_n
\end{aligned}$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project 1

Problem 1.1.
(a) Describe all possible solutions to the matrix equation system

$$x_1^2 = x_1, \quad x_2^2 = x_2, \quad x_1 x_2 = x_2 x_1 = 0$$

in two variables in M_2, up to conjugation by Aut_2.
(b) Describe all those conjugation classes that satisfy the additional equation

$$x_1 + x_2 = I_2.$$

(c) Describe all possible two-sided ideals of M_2.

We saw in class that the algebra M_n itself is an M_n-space on which M_n acts by left multiplication. We also saw that an F-subspace $W \subset M_n$ is an M_n-subspace iff W is a left ideal of M_n.

Problem 1.2.
(a) Describe all possible left ideals I of M_2. Which ones of them are isomorphic to each other?
(b) Classify all minimal M_2-spaces V up to isomorphisms.
(c) Classify all M_2-spaces V up to isomorphisms.

Problem 1.3. *Generalize both Problems 1.1 and 1.2 to M_n-spaces for all n.*

2. Statements of Problems in Project 2

All graphs are assumed finite (i.e. have finite number of vertices), planar (i.e. you can draw on the plane) and oriented (i.e. every edge has a direction). All F vector spaces considered here are finite dimensional. A face of a graph L is a free region (i.e. no edges crosses it) bounded by edges of L. Let L be a graph, and $\mathcal{V}_L$, $\mathcal{E}_L$, $\mathcal{F}_L$ be the sets of vertices, edges, and faces of L respectively. We drop L if there is no confusion.

Every face $\phi \in \mathcal{F}$ is given the counter-clockwise orientation. Note that we can label ϕ by giving the list of edges in counterclockwise order. This induces an orientation on each edge e of ϕ which may or may not agree with the *given* orientation of e. We assign $+1$ to e and write $(\pm)_{\phi,e} = +1$ if the orientation induced by ϕ on e is the same as the given orientation of e. Otherwise we assign $(\pm)_{\phi,e} = -1$. We can label each $e \in \mathcal{E}$ by its vertices, say (a, b), where a is the initial and b is the end vertices of e. For convenience, we treat $(b, a) \equiv -(a, b)$ for each oriented edge $(a, b) \in \mathcal{E}$.

This project is about studying certain connections between graphs and F-vector spaces. Given such a graph L, we shall form a collection of vector spaces as follows:

- $C_0(L)$ is the F vector space given by taking the set of vertices of L as a basis of $C_0(L)$, i.e. $C_0(L) = F\mathcal{V}$.
- $C_1(L)$ is the F vector space given by taking the set of edges of L as a basis of $C_1(L)$, i.e. $C_1(L) = F\mathcal{E}$.
- $C_2(L)$ is the F vector space given by taking the set of faces of L as a basis of $C_2(L)$, i.e. $C_2(L) = F\mathcal{F}$.

This collection $C_\bullet(L)$ of vector spaces is called the **complex** of L.

We define a collection of linear maps, we call the boundary maps of L, between these vector spaces as follows. Write $C_i = C_i(L)$. Define

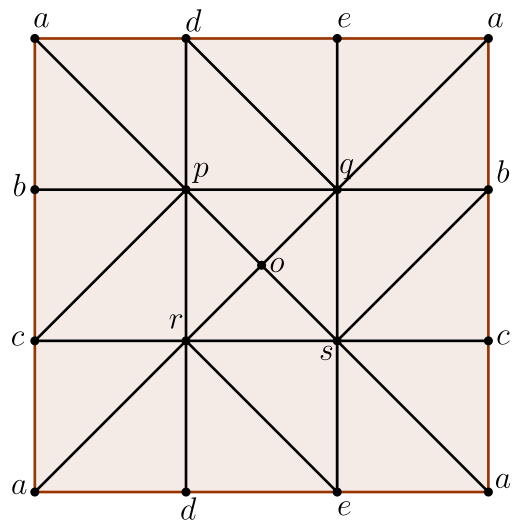
- $\partial_3 : (0) \to C_2$ is the zero map.
- $\partial_2 : C_2 \to C_1$, $\partial\phi = \sum(\pm)_{\phi,e} e$, where the sum is over all edges of ϕ.
- $\partial_1 : C_1 \to C_0$, $\partial(a, b) = a - b$.
- $\partial_0 : C_0 \to (0)$ is the zero map.

If there is no confusion, we shall drop all subscript from ∂.

Exercise 2.1. *Convince yourself that $\partial^2 = 0$, i.e. $\partial_i\partial_{i+1} = 0$ for all i. Therefore, we can define the* **homology** *spaces of L as*

$$H_i(L) := \ker \partial_i / \operatorname{im} \partial_{i+1}, \quad i = 0.1.2.$$

In this project, you are asked to determine the dimensions of the homology spaces of certain graphs. Consider the following graph, denoted by L.

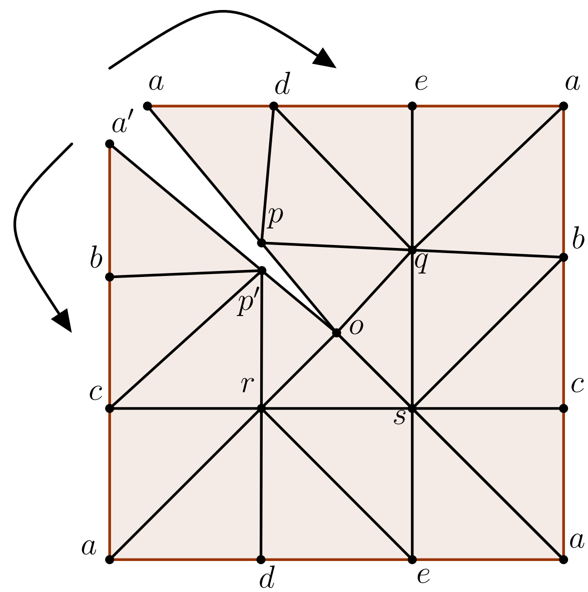


The boundaries of L (the brown edges) are identified via the labeling. We pick an orientation for every edge. For example, we can use the lexicographic ordering the vertex labels to give a edge an orientation: the edge with label (a, d) will have direction a to d, because a comes before d in the lexicographic ordering.

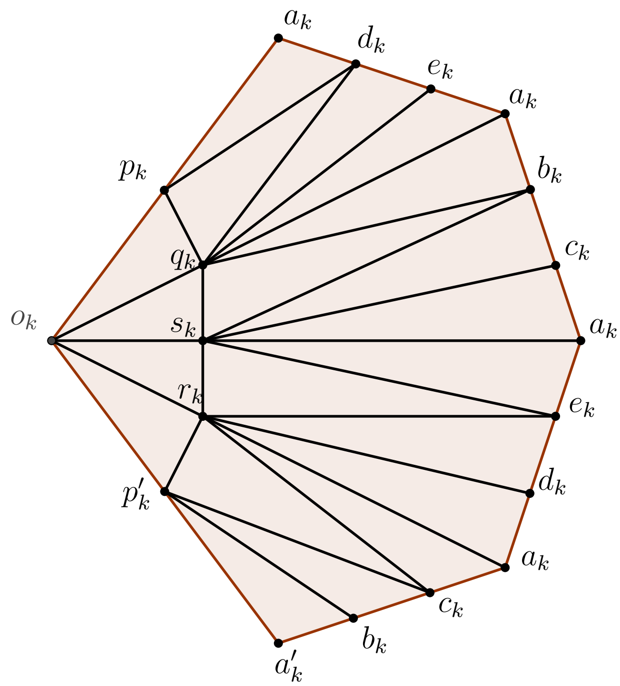
Problem 2.2. *Verify that*

- $\mathrm{H}_0(L) \simeq F$,
- $\mathrm{H}_1(L) \simeq F^2$,
- $\mathrm{H}_2(L) \simeq F$.

Next, we shall construct more complicated graphs by 'gluing' together copies of the graph L above. First, we 'cut open' L along the line segment $\overline{opa}$ in L, and then relabel the vertices as follows to get a new graph A_k.



Let A_k be the graph obtained by 'opening up' the shape along the arrows, and then assign the following new labels on the vertices.



For $n \in \mathbb{N}$, consider the set of graphs $\{A_1, \ldots, A_n\}$. Let S_n be the graph obtained by identifying $\overline{o_k p_k a_k}$ with $\overline{o_{k-1} p'_{k-1} a'_{k-1}}$ for $1 \leq k \leq n$. Here $\overline{o_0 p'_0 a'_0}$ is understood as $\overline{o_n p'_n a'_n}$. We orient each face in S_n counterclockwise and each edge in S_n, by lexicographic ordering as before. Again we get $C_\bullet(S_n)$ of the graph S_n. Note that $S_1 = L$.

Problem 2.3. *Find the homology spaces* $\mathrm{H}_q(S_n)$, *for all* n.

You should prove your answer.

3. Statements of Problems in Project 3

All vector spaces in this project are assumed finite dimensional.

A **complex** is a sequence of F-linear maps

$$\partial_{i+1} : C_{i+1} \to C_i, \quad i = 0, ..., d-1$$

such that the successive compositions are all zero: $\partial_i \partial_{i+1} = 0$. For convenience, we shall always assume that $C_i := (0)$, and that $\partial_{i+1} : C_{i+1} \to C_i$ are zero for all $i < 0$ or $i > d-1$. We denote the complex by $(C_\bullet, \partial)$ or simply $C_\bullet$, if ∂ is clear. The ith **homology** space of $C_\bullet$ is defined as the quotient spaces

$$H_i(C_\bullet) := \ker \partial_i / \operatorname{im} \partial_{i+1}, \quad i \in \mathbb{Z}.$$

A chain map between two complexes $(C_\bullet, \partial)$ and $(D_\bullet, \delta)$ is a collection of linear maps

$$f_i : C_i \to D_i$$

such that $\delta_i f_i = f_{i-1} \partial_i$ for all i. We shall denote the chain map by $f_\bullet : C_\bullet \to D_\bullet$.

We say that the two complexes are **equivalent** if there is a chain map $f_\bullet$ as above, such that each f_i is linear and bijective. We say that the two complexes are **quasi-equivalent** if there are two chain maps $f_\bullet : C_\bullet \to D_\bullet$, $g_\bullet : D_\bullet \to C_\bullet$, not necessarily bijective, such that their induced maps $\bar{f}_\bullet$, $\bar{g}_\bullet$ (to be defined in class) on homology are both linear and bijective, and are inverses of one another.

Problem 3.1. *Classify all equivalence classes of complexes $C_\bullet$ with at most two terms (i.e. all but two C_i are zero spaces). Do the same for 3-term complexes.*

Problem 3.2. *Classify all equivalence classes of complexes $C_\bullet$.*

Problem 3.3. *Classify all quasi-equivalence classes of complexes $C_\bullet$.*

4. Statements of Problems in Project 4

This is an open problem. You are encouraged to do some research online about the background of this problem. Be sure to give accurate references to whatever relevant information you have found.

Problem 4.1. *Given two commuting complex matrices A, B of size $n \times n$, when are they both polynomials of a third matrix X? That is to say, give a criterion such that there exists a pair of polynomials $p(t), q(t) \in \mathbb{C}[t]$ and a matrix X such that*
$$A = p(X), \quad B = q(X).$$

Problem 4.2. *Classify all pairs (A, B), up to conjugations by Aut_n, of complex $n \times n$ commuting matrices:*
$$AB = BA.$$